What is Napoleon’s Theorem?

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July 31, 2012

**Theorem 1 (Napoleon’s Theorem)**

If equilateral triangles are constructed on the sides of any triangle, all facing outward (resp. inward), then the triangle formed by connecting the centers of those equilateral triangles is also equilateral.

![Diagram of Napoleon's Theorem](image)

**Definitions**

\[ e^{i\theta} = \cos(\theta) + \sin(\theta)i \]

is a rotation by \( \theta \) counterclockwise.

The *center* of a triangle \( ABC \) is \( \frac{A + B + C}{3} \).

A matrix is called *circulant* if it has the form:

\[
\begin{bmatrix}
  a_1 & a_2 & \cdots & a_n \\
  a_n & a_1 & \cdots & a_{n-1} \\
  \vdots & \vdots & \ddots & \vdots \\
  a_2 & a_3 & \cdots & a_1
\end{bmatrix}
\]

It is written circ\((a_1, a_2, ..., a_n)\).
Related Results
1. Let \( z_1, z_2, z_3, z_4 \) be the vertices of a quadrilateral. Connect the midpoints cyclically. The resulting figure is always a parallelogram.

2. Consider the operator \( T_r = \text{circ}(1 - r, r, 0, ..., 0) \).

**Theorem 2**
Let \( n \geq 3 \) be an integer and \( r \in \mathbb{Q} \cap (0, 1) \). The following are equivalent:
(i) \( T_r \) is eventually periodic on all \( n \)-gons.
(ii) The operator \( T_r \) is periodic on some nondegenerate, nonregular \( n \)-gon \( Z \).
(iii) Either \( r = \frac{1}{2} \), or \( n = 3 \) and \( r = \frac{1}{3} \) or \( \frac{2}{3} \).

3. One can generalize Napoleon-like results to \( K_r \)-grams.
Let \( w = e^{\frac{2\pi i}{n}} \). A \( K_r \)-gram is defined as an \( n \)-gon satisfying \( K_r P = 0 \) where \( K_r = \text{circ}(1, w^r, w^{2r}, ..., w^{(n-1)r}) \).

Notice, if \( n = 3 \), then \( K_1 P = 0 \) iff \( P \) is equilateral.
And, if \( n = 4 \), then \( K_2 P = 0 \) iff \( P \) is a parallelogram. So we can think of \( K_r \)-grams as a generalization of such regularity conditions to \( n \)-gons.

**Theorem 3**
Let \( P \) be an \( n \)-gon and \( C \) a circulant matrix of rank \( n - 1 \). Then there exists an \( n \)-gon \( \hat{P} \) such that \( C \hat{P} = P \) if and only if \( P \) is a \( K_r \)-gram.

**Theorem 4**
Let \( C \) be a circulant matrix of rank \( n - 1 \). Then: \( P \) is a \( K_r \)-gram if and only if there is an \( n \)-gon \( Q \) such that \( P = CQ \).

References