What is Napoleon's Theorem?

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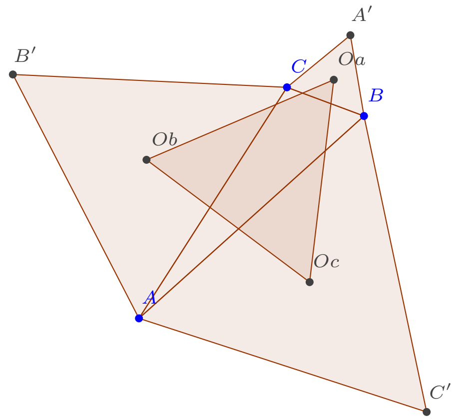
Theorem 1 (Napoleon's Theorem)
If equilateral triangles are constructed on the sides of any triangle, all facing outward (resp. inward), then the triangle formed by connecting the centers of those equilateral triangles is also equilateral.



Definitions
$e^{i\theta} = \cos(\theta) + \sin(\theta)i$ is a rotation by θ counterclockwise.
The *center* of a triangle ABC is $\dfrac{A+B+C}{3}$.
A matrix is called *circulant* if it has the form:

$$\begin{bmatrix} a_1 & a_2 & \cdots & a_n \\ a_n & a_1 & \cdots & a_{n-1} \\ \ddots & \ddots & \ddots & \ddots \\ a_2 & a_3 & \cdots & a_1 \end{bmatrix}$$

It is written $\mathrm{circ}(a_1, a_2, ..., a_n)$.

Related Results

1. Let z_1, z_2, z_3, z_4 be the vertices of a quadrilateral. Connect the midpoints cyclically. The resulting figure is always a parallelogram.

2. Consider the operator $T_r = \mathrm{circ}(1-r, r, 0, ..., 0)$.

Theorem 2

Let $n \geq 3$ be an integer and $r \in \mathbb{Q} \cap (0,1)$. The following are equivalent:
(i) T_r is eventually periodic on all n-gons.
(ii) The operator T_r is periodic on some nondegenerate, nonregular n-gon Z
(iii) Either $r = \frac{1}{2}$, or $n = 3$ and $r = \frac{1}{3}$ or $\frac{2}{3}$.

3. One can generalize Napoleon-like results to K_r-grams.
Let $w = e^{\frac{2\pi i}{n}}$. A K_r-gram is defined as an n-gon satisfying $K_r P = 0$ where $K_r = \mathrm{circ}(1, w^r, w^{2r}, ..., w^{(n-1)r})$
Notice, if $n = 3$, then $K_1 P = 0$ iff P is equilateral.
And, if $n = 4$, then $K_2 P = 0$ iff P is a parallelogram. So we can think of K_r-grams as a generalization of such regularity conditions to n-gons.

Theorem 3

Let P be an n-gon and C a circulant matrix of rank $n-1$. Then there exists an n-gon $\hat{P}$ such that $C\hat{P} = P$ if and only if P is a K_r-gram.

Theorem 4

Let C be a circulant matrix of rank $n-1$. Then: P is a K_r-gram if and only if there is an n-gon Q such that $P = CQ$.

References

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3. P Davis, Some Geometric Applications of Circulants. In: Circulant Matrices. New York: John Wiley & Sons, 1979, pp. 139-153.

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