WHAT IS NOETHER'S THEOREM?

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ABSTRACT. Noether's theorem states that given a physical system, for every infinitesimal symmetry, there is a corresponding law of symmetry. While this requires some parsing, it shows that the conservation of energy and momentum are mathematical consequences of facts that we take for granted: that the laws of physics do not depend on the time or location of the observer. We demonstrate the result in classical mechanics before exploring the theorem in more sophisticated settings.

1. INTRODUCTION AND NEWTONIAN MECHANICS

Most students are presented with Newton's Three Laws at a young age. They are as follows:

- An object at rest remains at rest unless acted upon by an outside force. An object in motion remains in motion unless acted upon by an outside force.
- The force is equal to the mass of the object times its acceleration.
- For every action there is an equal and opposite reaction.

Now that we are a bit older, we can translate these three laws into mathematical laws and try to make sense of them. So let's start with an object, and try to figure out what these laws say about its trajectory. We'll call the position of this object at time t $\gamma(t)$ and assume it has mass 1 because we might as well choose convenient units. The first two laws combined state that $\partial_t^2 \gamma = F$. Notice that in the absence of forces, we have $\partial_t^2 \gamma = 0$, which is exactly the First Law. The Third Law is a bit trickier to formulate, but it states that if an object exerts a force on another object, the latter object

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exerts an equal and opposite force against the former.[1] One way to think of this is that if we have a closed system of several objects on which no external forces act, then the total force on the system is zero since each force acting on each body in the system will cancel.

If we have a body, we can integrate all of the forces that act on it along the path it follows and call this quantity W, for work.

$$W = \int_a^b F \ ds$$

If the body is part of a closed system, then by the third law the work on an object is completely cancelled out by the work that the object exerts on the other bodies, so the total work on the system is zero. Since this work is a group over some "mysterious" torsor, this states that for whatever that torsor happens to be, it is conserved in a closed system. Even if you don't know what a torsor is, you are probably aware that work is a measurement of change of energy and so this shows that in a closed system, energy is a conserved quantity.

However, in this derivation, we have made a crucial assumption. We have assumed that that path integral for work makes sense along the path without factoring in the fact the particle is moving in time. We have assumed that a force applied at one point in time is as good as a force applied at any other point in time. This information is contained in Newton's second law, but in order for conservation of energy to hold, we crucially needed this, even though we completely skipped over it in the derivation. Therefore, in order for this to hold true, we needed the fact that the laws of physics are invariant in time.

One could also integrate all the forces over a path in time, and then we would derive conservation of momentum, with the essential assumption that a force applied at one point in space is as good as a force applied anywhere else, that is to say, the laws of physics are invariant of location.

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[1]Technically is states that the integrals of the forces cancel out, but under realistic conditions this is equivalent.

Both of these are examples of a much more general phenomena; for every conserved quantity there is an associated differentiable symmetry. This holds in great generality..

Exercise 1. *Find the associated symmetry associated to conservation of angular momentum and derive conservation of angular momentum from that symmetry.*

2. An Initial Proof and Lagrangian Mechanics

Unfortunately, in order to prove a simplified version of the theorem, Newtonian mechanics really aren't sufficient. Instead, one needs to consider Lagrangian mechanics. Lagrangian mechanics are a reformulation of classical mechanics that rely on Hamilton's of stationary action. However, for our purposes we do not need to understand how one would use them to solve for the dynamics of a physical system. They just provides the framework to prove the theorem. The following is a slight generalization of John Baez's blog post "Noether's Theorem in a Nutshell."

One starts with a function on the system, called the Lagrangian L. For classical mechanics, $L = T - V$ where T is the kinetic energy and V is the potential energy, but we can write $L = L(x, \dot{x}, t)$ in general where $\dot{x}$ is the time derivative of the particles in the system. Now, in Lagrangian mechanics, the momentum is defined to be $\partial_{\dot{x}} L$ and the force is defined to be $\partial_x L$. By the Euler-Lagrange equations, we know that the time derivative of momentum is the force, so $\partial_t(\partial_{\dot{x}} L) = \partial_x L$. A dot over a variable indicates differentiation with respect to time. Notice that I am being very vague about what space I am working on. This is not accidental. For the next few paragraphs, if you would like, you can imagine that you have a particle on the real line, x is its position, $\dot{x}$ is its velocities and ignore the superscript i's.

Exercise 2. *Convince yourself that these definitions make sense and agree with the the normal definitions of force and momentum when $L = T - V$ as in classical mechanics on the real line.*

More generally, you can imagine that $x = (x^i)$ are coordinates of your favorite smooth manifold M, that $(x, \dot{x}) = (x^i, \dot{x}^j)$ are coordinates on TM with the summation convention is in effect whenever an index is repeated. In that case, $L : TM \times \mathbb{R} \to \mathbb{R}$ is a function and there is an $\mathbb{R}$-action on TM whose integral curves satisfy the Euler-Lagrange equations.

Theorem 3 (Noether's Theorem–Simplified). *Suppose the Lagrangian has a time-independent differentiable symmetry, that is a smooth one-parameter variation $x(s)$ under which it is invariant. Then the quantity $C = (\partial_{\dot{x}^i} L) \partial_s x^i$ is conserved in time.*

Proof. Given a smooth variational symmetry of the Lagrangian $x(s)$ where $x(0) = x$, we know that

$$\partial_s L(x(s), \dot{x}(s), t) = 0.$$

Consider $\partial_t C = \partial_t((\partial_{\dot{x}^i} L) \partial_s x^i)$. By the product rule, we have that this is $\partial_s x^i \ \partial_t(\partial_{\dot{x}^i} L) + (\partial_{\dot{x}^i} L) \ \partial_t \partial_s x^i$. By time-independence of the symmetry and Euler-Lagrange equations, we can rewrite this as $\partial_s x^i \ \partial_{x^i} L + (\partial_{\dot{x}^i} L) \ \partial_s(\partial_t x^i)$. However, by the chain rule, this is simply $\partial_s L(x, \dot{x}, t)$ since the symmetry is time independent. But $\partial_s L(x(s), \dot{x}(s), t) = 0$ so $\partial_t C = 0$ and C is conserved.

$\square$

Noether's theorem holds in far greater generality and one does not need the time invariance of the symmetry. However, this proof has the advantage of being quite simple and this is already a very general result.

Exercise 4. *If the Lagrangian does not depend on time, use a similar argument to show that $H = \partial_{\dot{x}^i} L \dot{x}^i - L$ is conserved. Remember that just because the Lagrangian function does not depend on time, the total Lagrangian might because it depends on x and $\dot{x}$. What is the common name for H?*

3. Why Do We Care?

Although this is undoubtedly an interesting result, in classical mechanics there are only three types of symmetries and so this seems like an unnecessarily large amount of machinery to build to show the conservation of the

three classical quantities. The use becomes far more evident in theoretical physics. One starts with a more general version of the theorem, which allows us to consider fields instead of only particles. In many settings, it is possible to write down the Lagrangian (or Lagrangian density) of the system. [2] One then studies this system by trying to find symmetries of it. In most cases, one will obtain a Lie group called the "gauge group" of the system. By studying the associated Lie algebra of this gauge group, one finds the conserved quantities in the system. This is a standard tool for modern physicists and it has been said that Noether's theorem is second only to Pythagoras' theorem in importance for modern physicists. By using this in general relativity, one unifies the conservation of momentum and energy and shows why matter seems to be conserved on a day-to-day basis even though a priori there is no corresponding symmetry. However, the theorem really shines in particle physics and quantum mechanics.

Exercise 5. *What is the gauge group of the classical Lagrangian for a single free particle in $\mathbb{R}^3$? (Hint: Don't bother writing out the Lagrangian, just think of symmetries.)*

Most physicists believe that the best theory for particle physics is the standard model, and the recent discovery of the Higgs boson stands as an enormous success of this theory. The Lagrangian of this particle theory has internal symmetry $SU(3) \times SU(2) \times U(1)$ (a type of gauge symmetry), from which one can discover conserved quantities that are completely non-obvious. Much of our understanding of sub-atomic physics comes from understanding symmetries and Noether's theorem is the bridge between symmetries of a mathematical object and conserved quantities we can observe.

4. Phase space

In the classical case, we saw that momentum and position were linked by Noether's theorem and the same for energy and time. Every physicist

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[2]There is also a version of the theorem involving Hamiltonians, which has a beautifully simple proof. However, it requires some background in symplectic geometry so I will not cover it here. It can be found in John Lee's Introduction to Smooth Manifolds.

will immediately have a reflexive response to the previous statement and the connection is not a coincidence, although it is a case in which correlation is not causation.

If a translation in one coordinate preserves the Lagrangian (or equivalently the Hamiltonian, if you are familiar with Hamiltonian mechanics), then the corresponding coordinate momenta, or phase, will be conserved. Conversely, if a coordinate is conserved, its phase will leave the Lagrangian invariant.

These pairings occur because each of these coordinates is the phase of the other. The following few paragraphs explain how this occurs in quantum mechanics by taking the Fourier transform of the associated operators.

4.1. **Quantum Mechanics (optional).** The following is derived from David Griffith's excellent book Introduction to Quantum Mechanics. Suppose that instead of a set of discrete particles, one has entired the wild world of quantum mechanics and instead one has a wave function Ψ that evolves in time by Schrodinger's equation. When one tries to observe this wave function, one hits the function with an operator that spits out an expectation value, or what you expect the observation to be. As per Ehrenfest's theorem, these expectation values behave the same way a particle would classically, which is why classical mechanics works well on large scales. In the case where your space is the real line, the operator for position is $\int_{\mathbb{R}} x\Psi\bar{\Psi} \; dx$ and the operator for momentum is $\int_{\mathbb{R}} \frac{\hbar}{i}(\partial_x\Psi)\bar{\Psi} \; dx$.

Exercise 6. *Convince yourself that the position operator is reasonable and makes sense. If you have taken quantum mechanics, do the same with the momentum operator.*

If we were to take the Fourier transform of Ψ then one obtains a corresponding wave equation Φ in phase space. That is,

$$\Phi(p) = \int_{\mathbb{R}} \Psi(x)e^{-ipx}dx$$

. The Fourier inversion theorem states that for sufficiently nice Ψ and Φ, $\Psi(x) = \int_{\mathbb{R}} \Phi(p)e^{ipx}dp$ as well.

Working like physicists, we are going to assume that we have all the convergence we could ever want and that both Φ and Ψ are as differentiable as we could ever want. There are plenty of functional analysts here who can tell you exactly what hypotheses one needs and I highly encourage learning some functional analysis before applying Fourier transforms all over the place. Here we are more worried about intuition though, so the rigor is skipped.

However, if we notice that $xe^{ipx/\hbar} = -i\hbar\partial_p e^{ipx/\hbar}$ and vice-versa, then one sees that in phase space, the operator for position is $-\frac{\hbar}{i}\partial_p$ and the operator for momentum is p (where these are the quantities that we stick in the integral between the wave function and its conjugate). Therefore, position and momentum are Fourier transforms of each other (up to a sign). For this reason, the phase space is actually called momentum space by physicists; it wasn't by accident that the letter p was used in the Fourier transform. It also turns out that time and energy are theoretically Fourier transforms, though the details of a time operator is problematic in realistic settings.

Exercise 7. *Show that the operators behave as described.*

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