WHAT IS... PENROSE'S TILES?

DONALD ROBERTSON

1. Tilings

A subset of $\mathbb{R}^2$ is called a *tile* if it is homeomorphic to the closed unit ball $\{x \in \mathbb{R}^2 : |x| \leq 1\}$. A tile is shown in Figure 1.

Figure 1. A pentagonal tile in $\mathbb{R}^2$.

A *tiling* of a subset A of $\mathbb{R}^2$ is a countable set $\mathscr{T}$ of tiles in $\mathbb{R}^n$ such that:

(1) $\cup\mathscr{T} = A$,
(2) whenever S, T belong to $\mathscr{T}$ and are distinct we have $\operatorname{int} S \cap \operatorname{int} T = \varnothing$.

Part of a pentagonal tiling is shown in Figure 2. A *patch* is a tiling by finitely many tiles of a connected, simply connected subset of $\mathbb{R}^2$ which cannot be disconnected by removing one point. To say that a patch $\mathscr{A}$ belongs to a tiling $\mathscr{T}$ means that $\mathscr{A}$ is a subset of $\mathscr{T}$.

Given a tiling $\mathscr{T}$ we define an equivalence relation $\sim$ on $\mathscr{T}$ by isometries. A set of representatives for $\sim$ will be called a set of *protiles* for $\mathscr{T}$. If a set $\mathscr{P}$ of tiles is a set of protiles for some tiling $\mathscr{T}$ we say that $\mathscr{P}$ *admits* $\mathscr{T}$.

Let $\mathscr{T}$ be a tiling of $\mathbb{R}^2$. An isometry σ of $\mathbb{R}^2$ is called a *symmetry* of $\mathscr{T}$ if it maps every tile of $\mathscr{T}$ onto a tile of $\mathscr{T}$.

2. Polygonal tilings

The Greeks knew that of the regular polygons, only the triangle, the square, and the hexagon could tile the plane. Kepler found (Harmonices Mundi, 1619) that there are only eleven Archimedean tilings. (Tilings by regular polygons where every vertex is the same.) It seems that his work was forgotten for about 300 years. A reference to it appeared as a note added to Sommerville's 1905 determination of the eleven Archimedean tilings. Robin (1887) and Andreini (1907) also made this determination, all independently.

Which single polygons admit tilings of the plane? All triangles do. So do all quadrilaterals! It can be shown that the only possible candidates are triangles, quadrilaterals, pentagons and hexagons. In 1918, K. Reinhardt showed in his doctoral thesis (Frankfurt) that there are only three types of convex hexagon which tile the plane. (The types are characterized by equalities of certain edge lengths and certain relations amongst angles.) Reinhardt also found five types of pentagon which tile the plane, but did not give chase. R. B. Kershner took up the matter in 1967, finding three more types of pentagon that tile the plane. He believed that the eight types he knew of where the only ones, but no proof was given in his paper.

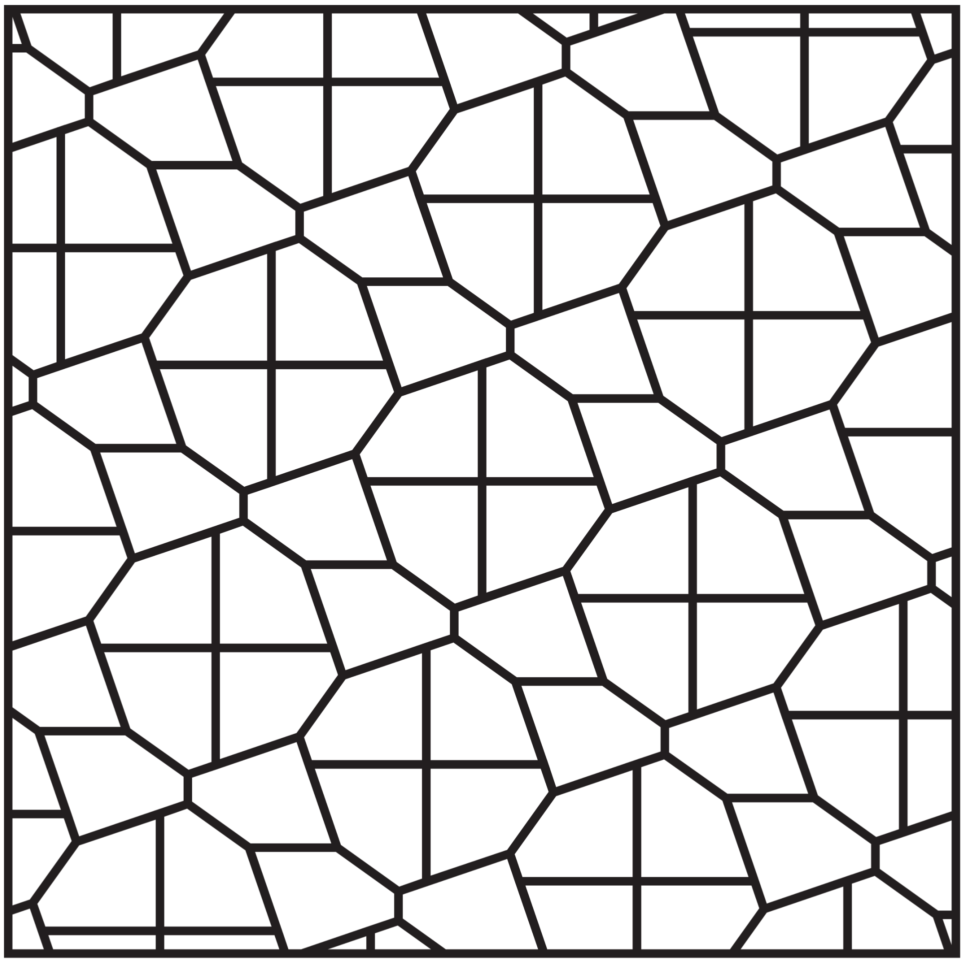

Figure 2. A tiling of the plane. Any single tile is a protile for this tiling. It can be seen in front of the elevators on the sixth floor of the math tower. A different pentagonal tiling can be seen outside the elevators on the seventh floor of the math tower.

According to the editor, this was "for the excellent reason that a complete proof would require a rather large book." This turned out to be the case: after Martin Gardner published an article about tilings by convex polygons in Scientific American, Richard E. James III wrote Gardner with a ninth type of pentagon which tiles the plane! In the 70s four more types were discovered by Marjorie Rice, a San Diego housewife. A fourteenth type was found in 1985 by Rolf Stein, a graduate student in Dortmund. No other types have since been found, and it is not known if the current list is complete.

3. Non-periodic tilings

A tiling $\mathscr{T}$ is *periodic* if it has at least two linearly dependent translation symmetries. Do non-periodic tilings exist? Certainly! We can take the square tiling and break its periodicity by splitting the squares in half, one vertically and the rest horizontally.

A more interesting question is whether there are tiles which admit tilings, all of which are non-peiodic. Such a set of tiles is called *aperiodic*. It turns out that this question is connected to the following question of decideability, called the *tiling problem*: Is there an algorithm for deciding whether a given set of tiles admits a tiling of the plane? This question was studied by Wang in the 60s and he broke sets of tiles into four collections:

(1) sets of tiles which admit no tiling;
(2) sets of tiles which only admit periodic tilings;
(3) sets of tiles which admit both periodic and non-periodic tilings;
(4) sets of tiles which only admit non-periodic tilings.

He was able to show that there is an algorithm which can distinguish the first three collections. That is, given a set of tiles $\mathscr{P}$ belonging to one of the first three collections, it is possible to decide to which collection it belongs. Wang then conjectured in 1961 that the fourth collection above is empty. Unfortunately for Wang's argument, in 1966 Berger discovered a set of 20,426 tiles which admit only non-periodic tilings. (In fact, the existance of such a set of tiles is equivalent to the decidability of the tiling problem.)

In 1973 and 1974 Roger Penrose (Emeritus Rouse Ball Professor of Mathematics at Oxford) discovered three sets of aperiodic tiles. We will work with the kite (K) and dart (D) tiles, shown in Figure 3. Only vertices of the same colour are allowed to meet. This is called a *matching condition*. It is equivalent to deforming the edges so that they can meet in only certain ways. Any tiling admitted by $\{D, K\}$ which adheres to the matching condition will be called a *Penrose tiling*.

The matching condition reduces the number of ways in which the tiles can meet at a vertex. For example, fitting four kites together at the acute black vertex is legal, but it is not legal to place a dart in the remaining space, even though there is room. There are seven ways in which the tiles can meet at a vertex. Such a meeting is called a *vertex neighbourhood*. A merry time can be had finding them all.

Whether there exists a single aperiodic tile is an open problem. Such a tiles does exist in space: the Conway biprism, discovered in 1993 by J. H. Conway. It fills space by layers placed on top of each other at an irrational angle.

4. Inflation and deflation

We can decompose a kite into two kites and two half-darts using the following procedure:

(1) Invert all vertex colours.
(2) Draw a line ℓ joining the two white vertices.
(3) Split both outer long edges of the kite in the ratio $1 : \tau$ using white vertices placed closer to the acute angle.
(4) Split the line ℓ in the ratio $1 : \tau$ using a black vertex, placing the vertex closer to the obtuse angle.

We can decompose a dart into one kite and two half-darts by the following procedure:

(1) Invert all vertex colours.

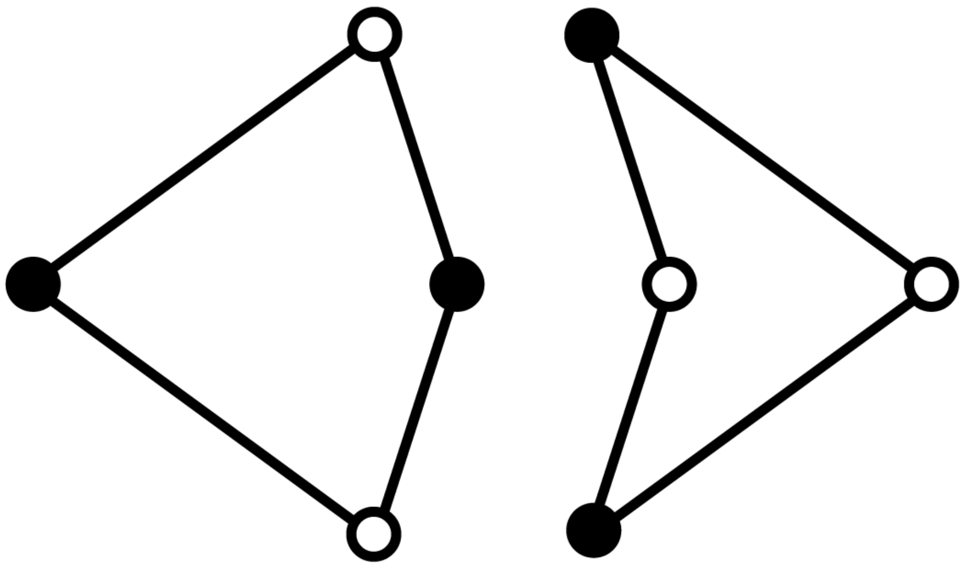

FIGURE 3. Penrose's kite (K) and dart (D) tiles. The length of the short edges is 1 and the length of the long edges is the golden ratio ϕ. The north, south, and west angles of the kite equal $2\pi/5$, as does the east angle of the dart. The black angles of the dart equal $\pi/5$.

(2) Split both long edges of the dart in the ratio $1 : \tau$ using white vertices placed closer to the white vertices.
(3) Draw lines between the new white vertices and the reflex vertex of the kite.

The above procedures are such that when adjacent tiles are decomposed, the half-darts join together to form darts in a way that preserves the matching condition. We call this *decomposition*. We denote the operation of decomposition by τ^{-1}. Thus given a Penrose tiling $\mathscr{T}$ its decomposition is another Penrose tiling $\tau^{-1}\mathscr{T}$. As the notation suggests, decomposition is reversible: given a Penrose tiling $\mathscr{T}$ we can fuse together kites and half-darts to make larger kites and darts in a unique way. We call this *composition* and denote this operation by τ, so that $\tau\mathscr{T}$ is the composition of $\mathscr{T}$.

The kites and darts which result from decomposition are smaller than those we start with. We can scale up the decomposition (by a factor of ϕ^2) to get kites and darts of the usual size. This process of decomposition followed by scaling up is called *inflation*. By *deflation* we mean the inverse procedure: first form the composition and then scale by a factor of ϕ^{-2}. We denote the nth inflation of a Penrose tiling $\mathscr{T}$ by $\mathscr{T}^{(-n)}$, and the nth deflation by $\mathscr{T}^{(n)}$.

We can start with a single kite and, by inflating again and again, tile arbitrarily large regions of the plane using kites and darts. This does not automatically show that we can tile the entire plane using kites and darts. To take this final step, we need the following extension theorem.

Theorem 1. *Let $\mathscr{P}$ be a finite set of tiles. If $\mathscr{P}$ admits tilings of arbitrarily large disks then $\mathscr{P}$ admits a tiling of the plane.*

Proof. See §3.8 of [4]. ☐

The theorem is not constructive: there is no connection between the tilings of the disks and the guaranteed tiling of the plane.

The fact that decomposition is invertible i.e., that every Penrose tiling has a unique composition, is key to the aperiodicity of $\{D, K\}$.

Theorem 2. *Let $\mathscr{T}$ be a Penrose tiling. Then $\mathscr{T}$ has no translational symmetry.*

Proof. Let $r \in \mathbb{R}^2$ be non-zero. Consider the sequence $(\tau^n \mathscr{T})$ of Penrose tilings obtained from $\mathscr{T}$ by repeated composition. The kites and darts get larger by a constant factor every time we compose, so there is a natural number N such that the kites and darts in $\tau^N \mathscr{T}$ contain disks of radius greater than $|r|$. Thus translation by r is not a symmetry of $\tau^N \mathscr{T}$.

Now suppose that translation by r is a symmetry of $\mathscr{T}$. Uniqueness of composition forces translation by r to be a symmetry of $\tau \mathscr{T}$ as well. Repeating this argument $N - 1$ times contradicts the previous paragraph. ☐

5. Examples

Two examples of Penrose tilings are the infinite sun and star patterns, which start from five kites and five darts respectively meeting at a vertex. Part of the star tiling is shown in Figure 4. The sun tiling (constructed using rhombs instead of kites and darts) can be seen in front of the elevators on the fifth floor of the math tower. If the pentagonal symmetry is preserved, there is exactly one way to legally continue the tiling and the plane can be tiled in this way. The sun and star tilings are the only Penrose tilings with a five-fold rotational symmetry, and they can be obtained from one another by inflation.

The cartwheel tiling is an important Penrose tiling, constructed as follows. Begin with the ace (Figure 5) and inflate it twice to get $\mathscr{C}_2$. Continuing this process gives the family $\{\mathscr{C}_{2n}\}$ of patches, called *cartwheels*. Each patch in the sequence embeds in the centre of the next patch, so

$$\mathscr{C} = \bigcup_{n=1}^{\infty} \mathscr{C}_{2n}$$

is a Penrose tiling, called the *Cartwheel tiling*.

6. Local Isomorphism

The following results are all from [4].

Theorem 3. *Every tile in the cartwheel tiling, except seven at the centre, lies within a cartweel having pentagonal symmetry.*

Proof. Induction on n. Inflating preserves pentagonal symmetries of patches, and every tile (excepting the exceptions) within $\mathscr{C}_4$ is contained in a patch with pentagonal symmetry. The theorem is proved by induction. ☐

Theorem 4. *For any natural n and any tile T in any Penrose tiling, there is a cartwheel $\mathscr{C}_{2n}$ in the tiling which contains T.*

Theorem 5. *In every Penrose tiling, all possible vertex neighbourhoods occur infinitely often.*

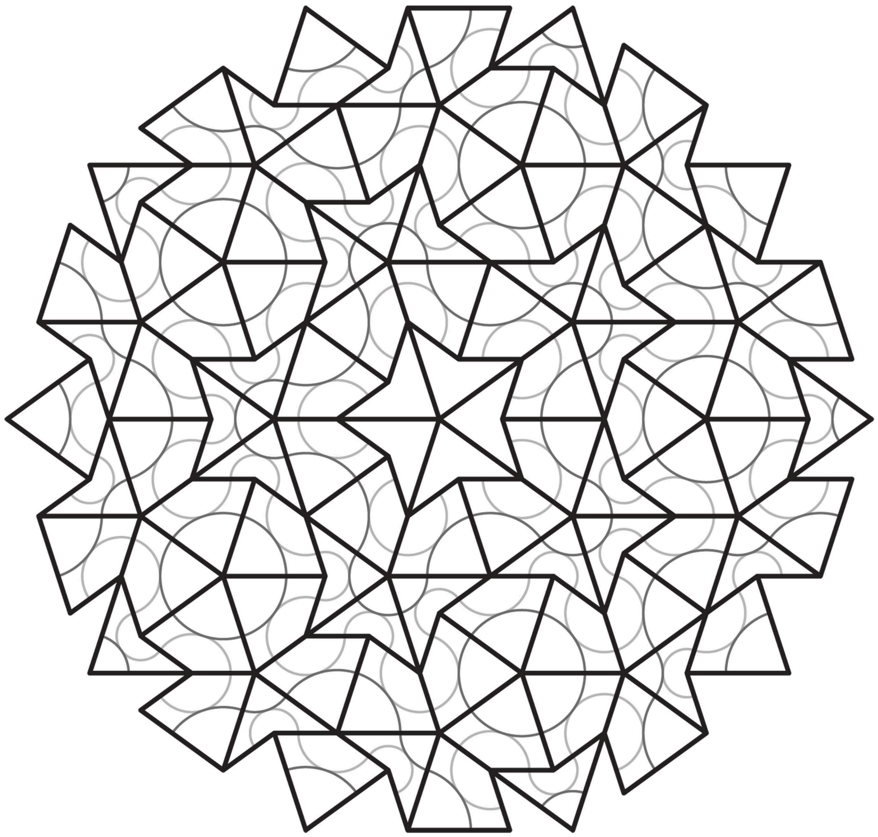

FIGURE 4. The start of the star tiling. (The light and dark curves give another way of drawing the matching conditions. Using this method, light and dark curves must meet at edges. It has been shown that whenever a curve closes, it will have pentagonal symmetry.)

Theorem 6. *Every patch $\mathscr{A}$ of tiles in a Penrose tiling $\mathscr{T}$ is congruent to infinitely many patches in every other Penrose tiling.*

Proof. Compose $\mathscr{T}$ until $\mathscr{A}$ contains at most one vertex of $\mathscr{T}^{(n)}$ and apply the previous theorem. □

The above theorem tells us that we cannot distinguish Penrose tilings from one another by looking at finie pieces. Thus Figure 4 is a picture of every Penrose tiling!

7. Further reading

Penrose introduced his tiles in [5]. It is possible that they had been used in mosque tilings much earlier, however. See [1].

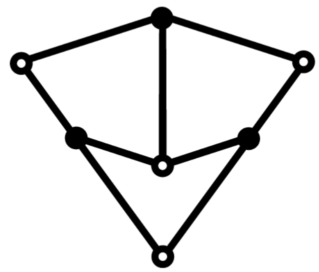

FIGURE 5. The ace $\mathscr{C}_0$.

If you want to know absolutely everything that was known about tilings in 1987, read [4]. Martin Gardner has written numerous articles about tilings and some can be found in his collections [2] and [3]. (Even if you are not interested in tilings, [3] is well worth reading. It contains many of Gardner's Scinetific American columns, updated and with addenda.) The book [8] contains many interesting topics on tiling, including obtaining aperiodic tilings as projections of higher-dimensional lattices.

Connections with quasicrystals (crystals which have sharp diffraction patterns but do not have crystallographic symmetry) are discussed in [8]. See also [9], [6] and [10].

One can study the dymanics of Penrose tilings. See [7] and the references therein.

References

[1] J. Bohannon, *Quasi-crystal conundrum opens a tiling can of worms*, Science, 315 (2007).

[2] M. Gardner, *Time Travel and Other Mathematical Bewilderments*, W. H. Freeman & Co., 1988.

[3] ———, *The Colossal Book of Mathematics*, W. W. Norton & Co., 2001.

[4] B. Grünbaum and G. C. Shephard, *Tilings and Patterns*, W. H. Freeman & Co., 1987.

[5] R. Penrose, *The role of aesthetics in pure and applied mathematical research*, Bull. Inst. Math. Appl., 10 (1974), pp. 166–271.

[6] C. Radin, *Quasicrystals and geometry*, Notices of the AMS, 43 (1996), pp. 416–421.

[7] E. A. Robinson, *The dynamical theory of tilings and quasicrystallography*, Ergodic Theory of $\mathbb{Z}^d$-Actions, LMS 228 (1996), pp. 451–473.

[8] M. Senechal, *Quasicrystals and Geometry*, Cambridge, 1995.

[9] ———, *What is... a quasicrystal?*, Notices of the AMS, 53 (2006), pp. 886–887.

[10] W. Steurer and S. Deloudi, *Fascinating quasicrystal*, Acta. Cryst., A64 (2008), pp. 1–11.

8. Acknowledgements

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