Ph.D. Algebra Qualifying Exam
6111

August, 2016
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Directions

1. Initial the roster sheet and enter a code name for yourself that is different from any code name that has already been entered.

2. Answer each question on a separate sheet or sheets of paper, and write your *code name* and the *problem number* on each sheet of paper that you submit for grading. Do not put your real name on any sheet of paper that you submit for grading.

3. Answer as many questions as you can. Do not use theorems which make the solution to the problem trivial. Always clearly display your reasoning. The judgment you use in this respect is an important part of the exam.

4. This is a closed book, closed notes exam.

1. Let G be a finite group which acts transitively on a finite set S with $|S| \geq 2$. Show that there there exists an element $g \in G$ which has no fixed points, i.e., for all $s \in S$, $g \cdot s \neq s$ (i.e., $S^{\langle g \rangle} = \emptyset$).

2. Given a group G, define the following normal subgroups inductively: $Z_0(G) = 1$, $Z_1(G) = Z(G)$, and $Z_{i+1}(G) \geq Z_i(G)$ the subgroup of G such that $Z_{i+1}(G)/Z_i(G) = Z(G/Z_i(G))$. The chain $Z_0(G) \leq Z_1(G) \leq Z_2(G) \leq \cdots$ is called the upper central series of G. A group is called nilpotent if $Z_c(G) = G$ for some integer c, and the smallest such c (if it exists) is called the nilpotence class of G.

   (a) Let p be a prime and suppose that G a group of order p^a where $a \geq 1$. Show that G is nilpotent.

   (b) If $a \geq 2$, prove that G is of nilpotence class $\leq a - 1$.

3. Let R be a commutative ring with 1 and M be an R–module. For a prime ideal $\mathfrak{p} \subset R$, let $M_{\mathfrak{p}} = S^{-1}M$ be the localization of M at the multiplicatively closed set $S = R - \mathfrak{p}$. Prove that

$$M = 0 \text{ if and only if } M_{\mathfrak{p}} = 0 \text{ for every prime ideal } \mathfrak{p}$$

4. Let R be a finite commutative ring with 1. Show that every prime ideal is maximal. (Hint: First show every finite integral domain is a field.)

5. Let D_8 be the dihedral group of order 8, so the symmetries of a square. Compute (with justification) the character table of D_8.