Phyllotaxis

phyllon = leaf
taxis = order / arrangement

def: the study of patterns created by the arrangement of plant leaves, florets, seeds, & other plant organs

Types (of patterns):

1. Whorled - 2 or more leaves grow from each node.
   - decussate / opposite : 2 leaves
   - alternating (leaves grow above gaps of previous node) or superposed (leaves grow directly above leaves of previous node)

2. Spiral - 1 leaf grows from each node with a constant divergence angle.

def: divergence angle = the smallest angle formed from the 2 line segments connecting 2 consecutive primordia with the center (apex)
   - relatively constant

def: primordium = the earliest stage of development of an organ ≈ lattice pt.
   - distichous : 180° divergence angle
     - common in grasses
   - when referring to spiral < genetic / parastichies

def: genetic spiral = a continuous line through consecutive born primordia from oldest to youngest by the shortest path

NOTE: study done by Davis & Davis (1981)
They recorded direction of genetic spirals in 71,640 palm trees in 42 regions

⇒ In the Northern Hemisphere left-handed spirals outnumbered right-handed. The # of left-handed significantly decreased from N to S and in the Southern-Hemisphere right-handed spirals outnumber left-handed

- orientation effected by latitude?

def: parastichies - noticable spiral pattern in a plant
def: family of parastichies - given any parastichy the corresponding family is the set of parastichies with the same pitch, going in the same direction

- parastichy pair - 2 families
- If a family has n parastichies it is called an n-parastichy
- a parastichy pair by an m-spiraled family and an n-spiraled family is denoted (m,n)

→ m, n known as secondary numbers

def: opposed pairs - the 2 parastichies are winding in opposite directions
def: visible opposed parastichy pair - an opposed pair (m,n) such that

a) every intersection of the 2 spirals occurs at a primordium

or

b) after constructing a lattice out of your pattern the triangle with verticies $0, m, \& n$ where no other point on the lattice are within the triangle or its boundries.

~~Bravis-~~ Bravis-Theorem (1837)
On a n-parastichy of a phyllotactic spiral pattern, the numbers on two adjcent primordia (numbered by consecutive terms on genetic spiral) differ by n.

When describing a spiral pattern 2 measurements are required for an accurate description
1. Divergence Angle
2. Plastochrome ratio

Def: Plastrochrome ratio, R - ratio of distances from the center of 2 conscientive primordia to the center
- $R > 1$
- due to irregularities to calculate R one should measure a number of pairs and average their ratios

In order to achieve meaningful mathematical relations in phyllotaxis we must make some assumptions:
1. spirals in each family are evenly spaced
2. these spirals are identical
3. The spirals going through the centers of the primordia are logorithmic
   - the angle formed from a pt's tangent and the segment joining it to the center is constant
4. The divergence angles and plastochrone ratios remain constant

NOTE: some biologist believe mathematical developement takes away from The Biology while others see it as tools that give us better insight.

Fibonacci Sequence in Phyllotaxis

It is common for secondary numbers m, n in parastichy pairs to be consecutive terms in the Fibonacci sequence
- examples: pinecones, sunflowers, apples, dasies,
- This occurs when the divergence angle $d = 360\left(\frac{1}{\tau^2}\right) \approx 137.5°$

$\tau = \frac{1+\sqrt{5}}{2}$

NOTE: Study done by Brousseau (1968) found only 74 out of 4290 cones (1.7%) diveated from the Fibonacci patterns
- those who don't follow Fibonacci pattern usually follow another sequence (ex Lucas Sequence)

Phyllotatic Fractions
- approximations of divergence angle

a. pick a pt A draw a line connecting it to the center
b. pick a pt. B near the line

phyllotatic fraction = $\frac{\text{\# turns from A to B}}{\text{\# pts pass including B not A}}$

- consecutive fractions oscillate around d

Fundamental Theorem of Phyllotaxis
Gennetal form - Jean 1988

Let (m,n) be a parastichy pair, where m & n are relatively prime in a system with a divergence angle d.
The following properties are equivalent:
(1) There exist unique intergers v, u s.t. $0 \le v < n$, $0 \le u < m$ and $|mv - nu| = 1$ with $d \in [\frac{u}{m}, \frac{v}{n}]$
(2) The parastichy pair (m,n) is visible and opposite

- est. the complete geometrical relationship b/w visible and opposed parastichy pairs (m,n) and the divergence angle d of a system

- proof based on fact that (m,n) is visible iff $|m(nd) - n(md)| = 1$

Although could not find the original proof. I found and equivalent proof at

www.math.smith.edu/phyllo/Research/index.html
- click on link to "Summetry of Plants" (Scott Hotton's thesis)

Algorithms relating d & (m, n) using $d = \frac{1}{\tau^2}$

① Computational algorithm: given d, all visible pairs obtained

a. determine a sequence A of fractions $\left(\frac{Kd}{K}\right)$ in lowest terms $k = 1, 2, \ldots$

- If $Kd = n + 0.5$ $n \in \mathbb{N}$ choose both n & $n+1$

$$\frac{(1(0.382))}{1} = \frac{0}{1} \qquad \frac{(2(0.382))}{2} = \frac{1}{2} \qquad \frac{(3(0.382))}{3} = \frac{1}{3} \qquad \frac{(4(0.382))}{4} = \frac{2}{5}$$

$$\rightarrow \frac{0}{1}, \frac{1}{2}, \frac{1}{3}, \frac{2}{5}, \frac{3}{8}, \frac{4}{11}, \frac{5}{12}, \frac{5}{13}, \frac{5}{14}, \frac{6}{17}, \frac{7}{18}, \frac{7}{19}, \frac{8}{21}, \ldots$$

b. order the fractions by increasing values by inserting them consectively b/w $\frac{0}{1}$ & $\frac{1}{2}$ alternating which side of the line if in the middle

- NOTE: If d rational 2 sequences of v.o. pairs

$$\frac{0}{1} \;\Big|\; \frac{1}{2}$$

$$\frac{0}{1} \quad \frac{1}{3} \;\Big|\; \frac{1}{2}$$

$$\frac{0}{1} \quad \frac{1}{3} \;\Big|\; \frac{2}{5} \quad \frac{1}{2}$$

$$\frac{0}{1} \quad \frac{1}{3} \;\Big|\; \frac{2}{5} \quad \frac{3}{7} \quad \frac{1}{2}$$

$$\frac{0}{1} \quad \frac{1}{3} \quad \frac{3}{8} \;\Big|\; \frac{2}{5} \quad \frac{3}{7} \quad \frac{1}{2}$$

$$\vdots$$

$$\frac{0}{1} \quad \frac{1}{3} \quad \frac{6}{17} \quad \frac{5}{14} \quad \frac{4}{11} \quad \frac{7}{19} \quad \frac{3}{8} \quad \frac{8}{21} \;\Big|\; \frac{5}{13} \quad \frac{7}{18} \quad \frac{2}{5} \quad \frac{5}{12} \quad \frac{3}{7} \quad \frac{1}{2}$$

c. consecutive denominators on either side of the dividor are the v. o. pairs

$(1, 2)$ $(3, 2)$ $(3, 5)$ $(8, 5)$ $(8, 13)$ $(21, 13)$

2. Contraction Algorithm: given a visible opposed pair made with secondary #s in a sequence of normal & anomalous phyllotaxis, an interval for d is obtained

def: contraction of the visible opposed parastichy pair (m, n)

$n > m \quad (m, n-m)$

$n < m \quad (m-n, n)$

→ this is also a v.o.p. pair

Ex. $(m, n) = (19, 31)$

We are looking for end pts w/ denominators 19 & 31

a. Take consecutive contractions until pair of form $J(t, t+1)$ or $(2t+1, 2t+3)$ is obtained

NOTE: if $J(t, t+1)$ form use interval $\left[\frac{1}{J(t+1)}, \frac{1}{Jt}\right]$

$(19, 31)$

$(19, 12)$

$(7, 12)$

$(7, 5) \Rightarrow t = 2 \quad (2(2)+1, 2(2)+3)$

$d \in \left[\frac{2}{5}, \frac{3}{7}\right] = \left[\frac{t}{2t+1}, \frac{t+1}{2t+3}\right]$

$\langle 2, 5, 7, 12, 19, 31, \ldots \rangle$

b. write interval as $\left[\frac{t}{2t+1}, \frac{t+1}{2t+3}\right]$

c. Keep taking Farey sums of end pts until dens 19, 31 are reached

$\left[\frac{2}{5}, \frac{3}{7}\right] \rightarrow \left[\frac{2+3}{5+7}, \frac{3}{7}\right] = \left[\frac{5}{12}, \frac{3}{7}\right] \rightarrow \left[\frac{5}{12}, \frac{3+5}{12+7}\right] = \left[\frac{5}{12}, \frac{8}{19}\right] \rightarrow \left[\frac{13}{31}, \frac{8}{19}\right] \approx [150.97^\circ, 151.58^\circ]$

actual $d = 151.14^\circ$ with $t = 2$

Phyllotaxis

TYPES:

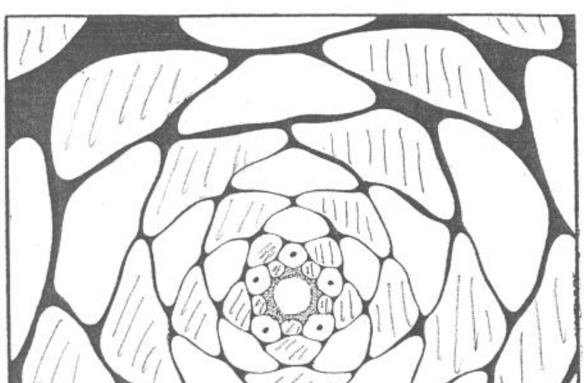
Whorled - more than 1 leaf per node

- Decussate / opposite : 2 leaves

Whorled → 3 leaves

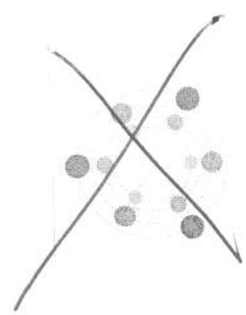
Decussate

- alternating
leaves are above the gaps of previous node



or

superposed
leaves are directly above leaves of previous node

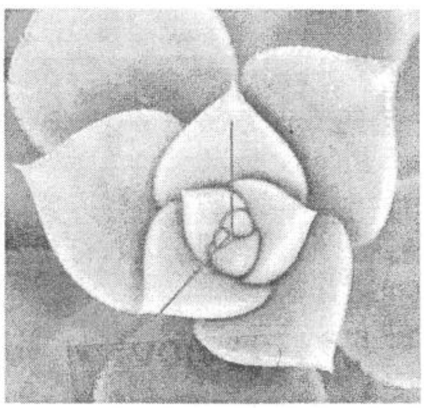


2. Spiral - one leaf grows from each node with a constant divergence angle

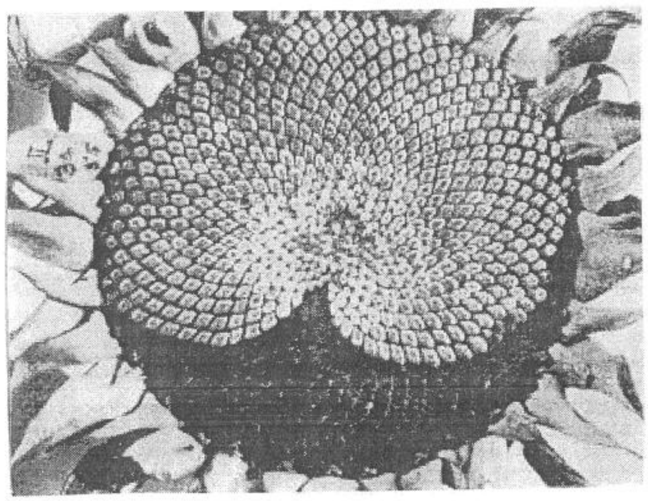
- distichous: 180° divergence angle

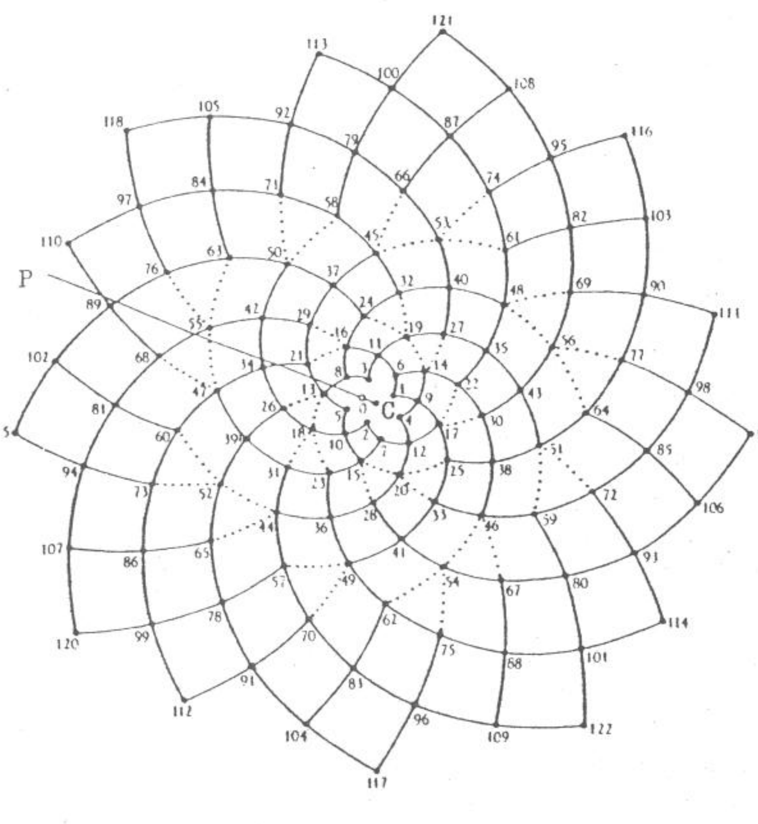
(b)

divergence angle



sunflower parastichy pair (34, 55)

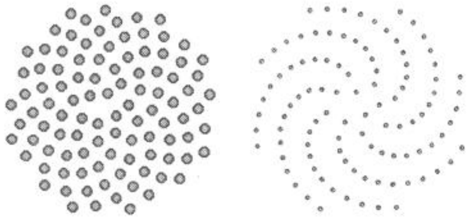




genetic spiral: 1-2-3-----121-122

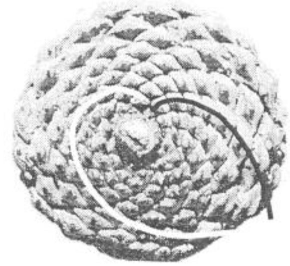
parastichies: drawn curves

family of parastichies

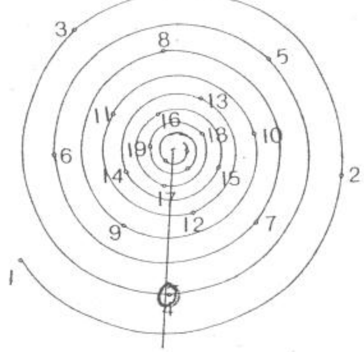


8- parastichy

pair of parastichies (opposed)



Phyllotactic Fractions



1 time around to get to 7 passed through 3 pts $\rightarrow \frac{1}{3}$

$\langle \frac{1}{3}, \frac{2}{5}, \frac{3}{8}, \frac{5}{13}, \ldots \rangle$

pts: 7 9 12 17

- This spiral was created with $d = \frac{1}{\tau^2}$

To make cylindrical lattice from pineapple-like strctures