What is Plateau's problem?

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1 Introduction

Plateau's problem is the problem of finding a minimal surface with a given boundary contour. It was first formulated by Lagrange (1760) in a rather restricted setting; Plateau showed by experiments that a minimal surface can be physically obtained in the form of a soap film stretched on a wire framework. (1849) Since then, it has been called the Plateau problem.

Important questions related to this problem includes, but are not limited to,

- existence of a solution
- uniqueness of the solution
- regularity of the solution

In this talk, I will focus on presenting minimal surface theory, which is at the heart of the problem, and some results on the existence of a solution to the problem. Brief accounts of achievements and a list of current open questions will conclude the talk.

2 Mathematics of Minimal Surfaces

Notation. Let D be an open set (usually an open disk or an open rectangle) in $\mathbb{R}^2$. The mapping $\mathbf{x} : D \to \mathbb{R}^3$, $\mathbf{x}\,(u, v) = \left(x^1\,(u, v)\,, x^2\,(u, v)\,, x^3\,(u, v)\right)$ is called a *parametrization* and its partial derivatives are denoted by

$$\mathbf{x}_u := \left(x_u^1, x_u^2, x_u^3\right), \qquad \mathbf{x}_v := \left(x_v^1, x_v^2, x_v^3\right).$$

Example 1. For a surface in $\mathbb{R}^3$ given by the graph of a two-variable function $z = f\,(x, y)$, we define a parametrization by $\mathbf{x}\,(u, v) = (u, v, f\,(u, v))$ where u and v range over the domain of f.

2.1 Definitions

Consider a surface M parametrized by $\mathbf{x}\,(u, v)$.

- A *unit normal vector* $\mathbf{N}$ to the tangent plane to M is given by

$$\mathbf{N} = \frac{\mathbf{x}_u \times \mathbf{x}_v}{|\mathbf{x}_u \times \mathbf{x}_v|}.$$

- A *normal curvature* $\kappa(\mathbf{w})$ in the tangent direction $\mathbf{w}$ is given by

$$\kappa(\mathbf{w}) = \mathbf{r}'' \cdot \mathbf{N}$$

where $\mathbf{r}$ is a unit speed parametrization of the intersecting curve of M and the plane spanned by $\mathbf{w}$ and $\mathbf{N}$. The derivative is taken along the curve $\mathbf{r}(s)$ with respect to s. By elementary vector calculus, we can write

$$\kappa(\mathbf{w}) = l\left(\frac{du}{ds}\right)^2 + 2m\frac{du}{ds}\frac{dv}{ds} + n\left(\frac{dv}{ds}\right)^2$$

where $l = -\mathbf{x}_u \cdot \mathbf{N}_u$, $2m = -(\mathbf{x}_v \cdot \mathbf{N}_u + \mathbf{x}_u \cdot \mathbf{N}_v)$, and $n = -\mathbf{x}_v \cdot \mathbf{N}_v$.

These l, m, and n are called the *coefficients of the second fundamental form.*

- $E = \mathbf{x}_u^2 = \mathbf{x}_u \cdot \mathbf{x}_u$, $F = \mathbf{x}_v^2 = \mathbf{x}_u \cdot \mathbf{x}_v$, and $G = \mathbf{x}_v \cdot \mathbf{x}_v$ are called the *coefficients of the metric.*

- The *mean curvature* (function) H is defined by

$$H = \frac{1}{2}(\kappa_1 + \kappa_2)$$

where κ_1 and κ_2 are the normal curvatures associated to any two perpendicular tangent vectors. Using the coefficients introduced above, we can write H as

$$H = \frac{En + Gl - 2Fm}{2(EG - F^2)}. \tag{2.1}$$

- A surface M parametrized by $\mathbf{x}(u, v)$ is said to be a *minimal surface* if $H = 0$ at each point on M.

Exercise 2. Let M be the graph of $z = f(x, y)$ with parametrization $\mathbf{x}(u, v) = (u, v, f(u, v))$. Using Formula (2.1), show that the condition for a minimal surface, $H = 0$, reduces to the partial differential equation

$$f_{uu}\left(1 + f_v^2\right) + f_{vv}\left(1 + f_u^2\right) - 2f_u f_v f_{uv} = 0.$$

This is known as the minimal surface equation.

2.2 Classical Examples

Classical examples of minimal surfaces are the plane, catenoid, and helicoid.

(1) Plane

(2) Catenoid: This is a surface of revolution generated by a catenary $y(x) = \cosh(x)$ and parametrized by

$$\mathbf{x}(u,v) = (u, \cosh(u)\cos(v), \cosh(u)\sin(v)).$$

(3) Helicoid: This is the shape of a spiral staircase whose parametrization is given by

$$\mathbf{x}(u,v) = (v\cos(u), v\sin(u), u).$$

It is a ruled surface, meaning that it is a trace of a line: for any point on the surface, there exists a line on the surface passing through it.

Exercise 3. Compute the mean curvature H of the catenoid and the helicoid and verify that they are indeed minimal surfaces.

It is known that:

Theorem 4. *(1) If a surface of revolution M is minimal, then M is contained in either a plane or a catenoid.*

(2) If a ruled surface M is minimal, then M is contained in either a plane or a helicoid.

Some named minimal surfaces are Enneper's surface, Henneberg's surface, Catalan's surface, and Scherk's Fifth surface, which will not be covered in this talk.

3 Plateau's Problem - Existence of a solution

Through his elaborate experiments, Plateau arrived at the conclusion that a (simple) closed curve, no matter how bizarre it is, always bounds a disk-like minimal surface. This is certainly a mathematical statement that a certain geometric boundary value problem always possesses a solution.

The precise mathematical formulation of Plateau's problem is as follows:

Problem 5. Given a Jordan curve (i.e. a simple closed curve) Γ in $\mathbb{R}^3$, determine a mapping $\mathbf{x} : \bar{D} \to \mathbb{R}^3$ where $D = \left\{(u,v) \in \mathbb{R}^2 \mid u^2 + v^2 < 1\right\}$ is the open unit disk and $\mathbf{x}(u,v) = \left(x^1(u,v), x^2(u,v), x^3(u,v)\right)$such that

(i) $\mathbf{x} \in C^2(D) \cap C^0\left(\bar{D}\right)$;

(ii) $\Delta\mathbf{x} = \mathbf{x}_{uu} + \mathbf{x}_{vv} = \mathbf{0}$, $E = \mathbf{x}_u^2 = \mathbf{x}_v^2 = G$, and $F = \mathbf{x}_u \cdot \mathbf{x}_v = 0$ on D;

(iii) $\mathbf{x}$ maps ∂D homeomorphically onto Γ.

The existence of a solution to this formulation was shown in 1930's by J. Douglas and T. Rado.

Theorem 6. *[Douglas, Rado] There exists a disk-like minimal surface spanning any given Jordan curve.*

For proofs, please look at references . We will only try to understand the formulation of this existence problem.

Any surface M parametrized by $\mathbf{x}$ satisfying (i), (ii), and (iii) is called a solution to Plateau's problem. Suppose M is one such surface. The condition that $\mathbf{x}$ is defined on $\bar{D}$ means that M is disk-like; Condition (i) allows us to deal with the derivatives in (ii); Condition (iii) simply means that M is bounded by the given curve Γ. So it remains to understand the connection between Condition (ii) and M being a minimal surface.

Definition 7. A parametrization $\mathbf{x}(u,v)$ is called *isothermal* if $E = G$ and $F = 0$.

Theorem 8. *[Osserman] If the parametrization* $\mathbf{x}(u,v)$ *is isothermal, then* $\Delta\mathbf{x} := \mathbf{x}_{uu} + \mathbf{x}_{vv} = (2EH)\,\mathbf{N}$.

Exercise 9. Prove Theorem 8.

Hint. Use Formula (2.1) and the following formulas:

$$\begin{aligned}
\mathbf{x}_{uu} &= \frac{E_u}{2E}\mathbf{x}_u - \frac{E_v}{2G}\mathbf{x}_v + lU \\
\mathbf{x}_{vv} &= -\frac{G_u}{2E}\mathbf{x}_u + \frac{G_v}{2G}\mathbf{x}_v + nU.
\end{aligned}$$

Corollary 10. *A surface M with an isothermal parametrization* $\mathbf{x}(u,v) = \left(x^1(u,v), x^2(u,v), x^3(u,v)\right)$ *is minimal if and only if* x^1, x^2, *and* x^3 *are harmonic functions.*

Proof. Suppose M is a minimal surface. Then $H = 0$ and so, by Theorem 8, $\Delta\mathbf{x} = \mathbf{0}$, which means that $\Delta x^i = 0$ for $i = 1,2,3$. Conversely, if x^i are harmonic, then $\Delta\mathbf{x} = \mathbf{0}$. By Theorem 8, $(2EH)\,\mathbf{N} = \mathbf{0}$. But $|\mathbf{N}| = 1$ and $E = \mathbf{x}_u^2 \neq 0$. So it must be the case that $H = 0$, meaning that M is minimal. □

So Condition (ii) is precisely the condition that M is a minimal surface.

4 Achievements and Open Problems

In early 1960's, several extensions of the problem to higher dimensions (i.e. for k-dimensional surfaces in n-dimensional space for $k \geq 3$ and $n \geq k$) were proposed based on new definitions of the concept of a surface, a boundary, and area. They turn out to be much more difficult to study. Moreover, while the solutions to the original problem are always regular, it turns out that the solutions to the extended problem may have singularities if $k \leq n-2$. For structures of codimension 1, i.e. for k-dimenional surfaces in $n = k+1$-dimensional space, singularities occur only for $n > 7$. To solve the extended problem, the theory of perimeters (De Giorgi) for codimension 1 and the theory of rectifiable currents (Federer and Fleming) for higher codimension have been developed.

As regards to uniqueness, only certain sufficient criteria are known, one of which is that the solution is unique if the given contour Γ has a single-valued convex projection under central or parallel projection onto a certain plane.

Current open problems abounds, some of which are

- Surfaces of least area which are forced to lie on one side of a fixed obstacle.
- Free boundary value problems where the boundary of a solution surface is requried to lie on a given manifold.
- The close relations to the calculus of variations and to the theory of partial differential equations.
- The behavior of interfaces under varying gravity conditions.
- The (non)existence problems for surfaces of prescribed (but variable and not vanishing) mean curvature.

References

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