

Department of Mathematics
Analysis Qualifying Exams
(past exams)

Please start each problem on a new page and remember to write your code on each page of your answers.

To pass, it is sufficient to solve four problems completely. You should exercise good judgment in deciding what constitutes an adequate solution. In particular, you should not try to solve a problem by just quoting a theorem that reduces what you are asked to prove to a triviality. If you are not sure whether you may use a particular theorem, ask the proctor.

1. Let $a_n \geq 0$ for $n = 1, 2, 3, \dots$ and suppose that $\sum_{n=1}^{\infty} a_n < \infty$. Prove that there exists a sequence $0 < b_1 < b_2 < \dots$ of real numbers such that $b_n \rightarrow \infty$ and $\sum_{n=1}^{\infty} a_n b_n < \infty$.
2. Let z be a complex number such that $|z| = 1$ but $z \neq 1$. Let $a_1 \geq a_2 \geq a_3 \geq \dots \geq 0$ and suppose $a_n \rightarrow 0$ as $n \rightarrow \infty$. Prove that the series

$$\sum_{n=1}^{\infty} a_n z^n$$

converges. (Don't just deduce this from a more general theorem. Give a detailed proof.)

3. Suppose that $f: \mathbf{R} \rightarrow [0, \infty)$ is twice continuously differentiable. Let K be the support of f . In other words, let K be the closure of $\{x \in \mathbf{R} : f(x) \neq 0\}$. Suppose that K is compact. Prove that there is a constant C (depending on f) such that for each $x \in \mathbf{R}$, we have

$$f'(x)^2 \leq C f(x).$$

4. Define a sequence (p_n) of polynomials $p_n: [0, 1] \rightarrow \mathbf{R}$ recursively as follows: For each $x \in [0, 1]$, let $p_0(x) = x$ and if $p_0(x), \dots, p_n(x)$ have already been defined, let

$$p_{n+1}(x) = p_n(x) + \frac{x - p_n(x)^2}{2}.$$

Prove that as $n \rightarrow \infty$, $p_n(x) \uparrow \sqrt{x}$ uniformly on $[0, 1]$.

5. Let $f: [0, \infty) \rightarrow \mathbf{R}$ be bounded and continuous. Prove that

$$\limsup_{b \rightarrow \infty} \frac{1}{b} \int_0^b f(x) dx \leq \limsup_{x \rightarrow \infty} f(x).$$

6. Let \mathcal{R} be the real vector space of Riemann integrable functions $f: [0, 1] \rightarrow \mathbf{R}$. For each $f \in \mathcal{R}$, let $\|f\| = (\int_0^1 |f(x)|^2 dx)^{1/2}$. Let $f \in \mathcal{R}$ and let $\varepsilon > 0$. Prove that there is a continuous function $g: [0, 1] \rightarrow \mathbf{R}$ such that $\|f - g\| < \varepsilon$.

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1. Prove that the sequence $(1 + \frac{1}{n})^{n^2} e^{-n}$, $n \in \mathbb{N}$, converges and find its limit.
2. Prove or disprove that the function $f(x) = \sin(x^3)/x$, $x > 0$, is uniformly continuous on $(0, \infty)$.
3. Let $f: \mathbb{R} \rightarrow \mathbb{R}$ be a twice-differentiable function with $f''(x) > 0$ for all $x \in [0, 1]$. Assume that $f(0) > 0$ and $f(1) = 1$. Prove that there exists $x_0 \in (0, 1)$ such that $f(x_0) = x_0$ if and only if $f'(1) > 1$.
4. Prove that $\sup_{x \geq 0} x e^{x^2} \int_x^\infty e^{-t^2} dt = \frac{1}{2}$.
5. If f is a Riemann integrable function on a closed bounded interval $[a, b]$, prove that $\lim_{n \rightarrow \infty} \int_a^b f(x) \cos^n x \, dx = 0$.
6. Prove that the series $\sum_{n=1}^{\infty} \frac{\sin nx}{n}$, $x \in [0, 1]$, does not converge uniformly on $[0, 1]$.

Please: Use only one side of the sheet, don't write on the other side. Begin each problem on a new sheet, and remember to write your secret **code** name and the problem number at the top of each page.

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1. Study the convergence of the sequence:

$$\sqrt{2}, \sqrt{1+\sqrt{2}}, \sqrt{1+\sqrt{1+\sqrt{2}}}, \sqrt{1+\sqrt{1+\sqrt{1+\sqrt{2}}}}, \dots$$

More precisely, decide if it is divergent (in this case, does it have an infinite limit?), or convergent (in which case find the limit if possible, otherwise estimate it).

2. Let $f: I \rightarrow \mathbf{R}$ be continuous and satisfy the inequality

$$f\left(\frac{x+y}{2}\right) \leq \frac{1}{2}(f(x) + f(y))$$

for all $x, y \in I$, where I is an interval in \mathbf{R} . Prove that f is convex. In other words, prove that

$$f((1-t)x + ty) \leq (1-t)f(x) + tf(y)$$

for all $x, y \in I$ and for all $t \in [0, 1]$.

3. Let $f: (1, \infty) \rightarrow \mathbf{R}$ be differentiable and define $g, h: (1, \infty) \rightarrow \mathbf{R}$ by

$$g(x) = \frac{f'(x)}{x} \quad \text{and} \quad h(x) = \frac{f(x)}{x}.$$

Suppose g is bounded. Prove that h is uniformly continuous.

4. Let $[a_1, b_1], [a_2, b_2], \dots, [a_n, b_n]$ be subintervals of $[a, b]$. Assume that each point x in $[a, b]$ lies in at least q of these subsets. Prove that there exists $k \in \{1, \dots, n\}$ such that $(b_k - a_k) \geq (b - a) \frac{q}{n}$.
5. Let $f(x) = \sum_{n=0}^{\infty} 2^n \sin \frac{1}{3^n x}$ for all $x > 0$ for which the series converges. Prove that f is defined and is differentiable on $(0, \infty)$.
6. Prove that $\sup_{x>0} x \int_0^{\infty} \frac{e^{-px}}{p+1} dp = 1$.

Ombudsman - Barbara Keyfitz

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Notation. \mathbf{R} denotes the set of real numbers and \mathbf{N} denotes the set $\{1, 2, 3, \dots\}$ of natural numbers.

1. Let f , g , and h be real-valued functions which are continuous on $[a, b]$ and differentiable on (a, b) , where $a, b \in \mathbf{R}$ with $a < b$. Define F on $[a, b]$ by

$$F(x) = \det \begin{pmatrix} f(x) & g(x) & h(x) \\ f(a) & g(a) & h(a) \\ f(b) & g(b) & h(b) \end{pmatrix}.$$

Prove that there exists $c \in (a, b)$ such that $F'(c) = 0$.

2. Prove that $e^\pi > \pi^e$.

3. Let $f: [1, \infty) \rightarrow \mathbf{R}$ be bounded and continuous. Prove that

$$\lim_{n \rightarrow \infty} \int_1^\infty f(t) n t^{-n-1} dt = f(1).$$

4. Let $f: \mathbf{R} \rightarrow \mathbf{R}$ be monotone and satisfy $f(x_1 + x_2) = f(x_1) + f(x_2)$ for all x_1 and x_2 in \mathbf{R} . Prove that $f(x) = ax$ for all real numbers x , where $a = f(1)$.

5. Let $\{r_k\}_{k=1}^\infty$ be the set of rational numbers of the interval $[0, 1]$. Define $f: [0, 1] \rightarrow \mathbf{R}$ by

$$f(x) = \sum_{k=1}^{\infty} \frac{|x - r_k|}{3^k}.$$

Then f is continuous on $[0, 1]$. (You may take this for granted.) Prove that f is differentiable at every irrational point in $(0, 1)$.

6. Consider a function $f(x) = \sum_{n=0}^{\infty} a_n x^n$ defined by a power series with radius of convergence $R \in (0, \infty)$. Suppose the series converges at $x = R$. Prove that f is left-continuous at $x = R$. (Of course this is commonly known as Abel's theorem on endpoint behaviour of power series. You are being asked to prove that theorem, not just quote it. Warning: The convergence at $x = R$ may be only conditional. Indeed, the result is almost trivial when the convergence there is absolute.)

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Prove that there exists $c \in (a, b)$ such that $F'(c) = 0$.

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Analysis Qualifying Examination

Spring, 2011

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1. Let $\lambda_1 < \lambda_2 < \lambda_3 < \dots$ be the consecutive strictly positive solutions of the equation $x = \tan x$. Does the series $\sum_{n=1}^{\infty} \lambda_n^{-2}$ converge? Justify your answer. (As part of justifying your answer, you should justify any estimates on λ_n that you use.)

2. Let $f: (0, \infty) \rightarrow \mathbf{R}$ be twice differentiable and suppose that $A, C \in [0, \infty)$ such that for each $x > 0$, we have $|f(x)| \leq A$ and $|f''(x)| \leq C$. Prove that for each $x > 0$ and each $h > 0$, we have

$$|f'(x)| \leq \frac{A}{h} + Ch.$$

3. Find the least constant c such that

$$(x_1 + x_2 + \dots + x_{2011})^2 \leq c(x_1^2 + x_2^2 + \dots + x_{2011}^2)$$

for all real values of $x_1, x_2, \dots, x_{2011}$. (For emphasis, let's repeat that you are asked to find the least such c , not just some such c .)

4. For each $x > 0$, the integral

$$I(x) = \int_0^{\infty} \frac{\sin xt}{1+t} dt$$

exists as a conditionally convergent improper Riemann integral. (You may take this fact for granted.) Prove that $I(x)$ has a limit in \mathbf{R} as $x \rightarrow 0+$.

5. Let (a_k) be a sequence of non-negative real numbers. Suppose that for each sequence (x_k) of non-negative real numbers with $\lim_{k \rightarrow \infty} x_k = 0$, the series $\sum_{k=1}^{\infty} a_k x_k$ converges. Prove that the series $\sum_{k=1}^{\infty} a_k$ converges.

6. For $n = 1, 2, 3, \dots$, define $f_n: [0, 1] \rightarrow \mathbf{R}$ by $f_n(x) = \frac{2n^2 x}{e^{n^2 x}}$. Does the sequence (f_n) converge uniformly on $[0, 1]$? Justify your answer.

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1. Let $S_n = \sqrt{1} + \sqrt{2} + \cdots + \sqrt{n}$. Find $\lim_{n \rightarrow \infty} \frac{S_n}{n^{3/2}}$. Justify your answer.
2. Let (a_n) be a sequence of real numbers such that the series $\sum_{n=1}^{\infty} a_n$ converges. Prove that

$$\frac{1}{n} \sum_{k=1}^n k a_k \rightarrow 0$$

as $n \rightarrow \infty$. (Hint: For any sequence (b_n) of real numbers, if $b_n \rightarrow b \in \mathbf{R}$ as $n \rightarrow \infty$, then $\frac{1}{n} \sum_{k=1}^n b_k \rightarrow b$ as $n \rightarrow \infty$. You may use this fact without proof.)

3. Let $f: [0, 1] \rightarrow \mathbf{R}$ be continuous. Suppose f is twice-differentiable on the open interval $(0, 1)$ and M is a real constant such that for each $x \in (0, 1)$, we have $|f''(x)| \leq M$. Let $a \in (0, 1)$. Prove that

$$|f'(a)| \leq |f(1) - f(0)| + \frac{M}{2}.$$

4. Let $f: [0, 1] \times \mathbf{R} \rightarrow \mathbf{R}$ be continuous, define $g: \mathbf{R} \rightarrow \mathbf{R}$ by

$$g(y) = \int_0^1 f(x, y) dx,$$

and suppose $\partial f / \partial y$ is continuous on $[0, 1] \times \mathbf{R}$. Prove that g is differentiable on \mathbf{R} and that

$$g'(y) = \int_0^1 \frac{\partial f}{\partial y}(x, y) dx$$

for all $y \in \mathbf{R}$.

5. Let X be a subgroup of $(\mathbf{R}, +)$. (This means that $0 \in X \subseteq \mathbf{R}$ and for all $x, y \in X$, we have $x + y \in X$ and $-x \in X$.) Prove that either X is dense in \mathbf{R} or there exists $c \in \mathbf{R}$ such that $X = c\mathbf{Z}$, where $c\mathbf{Z} = \{ck : k \in \mathbf{Z}\}$.
6. Let $f: [0, 1] \rightarrow \mathbf{R}$ be Riemann-integrable. For each real number $p > 1$, let

$$A_p = \int_0^1 p x^{p-1} f(x) dx.$$

Let $A = \limsup_{p \rightarrow \infty} A_p$ and let $B = \limsup_{x \rightarrow 1-} f(x)$. Prove that $A \leq B$. (In case you would like a reminder, here is one way to define the limits superior that appear in this problem: $\limsup_{p \rightarrow \infty} A_p = \inf_{q \in (1, \infty)} \sup_{p \in (q, \infty)} A_p$ and $\limsup_{x \rightarrow 1-} f(x) = \inf_{u \in (0, 1)} \sup_{x \in (u, 1)} f(x)$.)

Analysis Qualifying Examination

Spring 2010

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1. Prove that if $\{a_n\}$ and $\{b_n\}$ are convergent sequences of real numbers, with $A = \lim_{n \rightarrow \infty} a_n$ and $B = \lim_{n \rightarrow \infty} b_n$, then

$$\lim_{n \rightarrow \infty} \frac{a_0 b_n + a_1 b_{n-1} + \cdots + a_{n-1} b_1 + a_n b_0}{n+1} = AB.$$

2. For each $n \in \mathbb{N}$ let $f_n(x) = \frac{nx}{x^2 + n^2}$, $x \in \mathbb{R}$. Check whether the sequence (f_n) converges uniformly on \mathbb{R} .

3. Prove the following special case of the Riemann-Lebesgue lemma: Let $f : [0, 1] \rightarrow \mathbb{R}$ be continuous. Prove that $\lim_{t \rightarrow \infty} \int_0^1 f(x) \sin tx \, dx = 0$.

4. Prove this version of Cauchy's mean value theorem:

Let $f, g : [a, b] \rightarrow \mathbb{R}$ be continuous on $[a, b]$ and differentiable on (a, b) . Then there exists $c \in (a, b)$ such that

$$(f(b) - f(a))g'(c) = (g(b) - g(a))f'(c)$$

5. Find the limit

$$\lim_{z \rightarrow 0^+} \frac{1}{\ln z} \int_0^1 \frac{\cos t}{z+t} dt$$

6. Show that for all $x \in \mathbb{R}$ and all $n \in \mathbb{N}$ we have

$$e^x \geq \sum_{j=0}^{2n+1} \frac{x^j}{j!}$$

Please: Write your solutions only on one side of the sheet, inside the frame. Write solutions to distinct problems on distinct sheets. On the top of each sheet write your code (secret name), the number of the problem, and, if your solution to this problem occupies several pages, the number of the page of the solution.

Analysis Qualifying Exam

Wednesday, September 23, 2009

Kenneth Koenig, Ombudsman

1. For any sequence (a_n) of positive numbers, prove that

$$\limsup_{n \rightarrow \infty} (a_n)^{1/n} \leq \limsup_{n \rightarrow \infty} \frac{a_{n+1}}{a_n}.$$

2. Prove or disprove: $f(x) = x \log x$ is uniformly continuous on

- (i) the interval $(0, 1]$.
- (ii) the interval $[1, \infty)$.

3. Let $f : [0, 1] \rightarrow \mathbb{R}$ be continuously differentiable and satisfy $f(0) = f(1) = 0$. Show that

$$\int_0^1 |f(x)|^2 dx \leq 4 \int_0^1 x^2 |f'(x)|^2 dx.$$

4. Suppose we define the sine function by the convergent series

$$\sin x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!}.$$

Prove that there exists $a > 0$ such that $\sin(a) = 0$.

5. Prove that

$$\lim_{n \rightarrow \infty} \frac{\frac{2}{3}n^{3/2} - \sum_{k=1}^n k^{1/2}}{n^{1/2}}$$

exists and determine its value.

6. Let $f \in C([-1, 1])$ and $\int_{-1}^1 x^{2n} f(x) dx = 0$ for all integers $n \geq 0$. Prove that f is an odd function.

Analysis Qualifying Examination

Spring, 2009

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1. Let a_1 and b_1 be positive real numbers. Define

$$a_n = \sqrt{a_{n-1}b_{n-1}}, \quad b_n = \frac{1}{2}(a_{n-1} + b_{n-1})$$

for integers $n > 1$. Prove that both the sequences (a_n) and (b_n) converge and have the same limit.

2. Prove or disprove: If f is a continuous function on $[0, \infty)$ such that $\lim_{x \rightarrow +\infty} \frac{f(x)}{x} = 1$, then f is uniformly continuous on $[0, \infty)$.

3. Given that $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^2} = \frac{\pi^2}{12}$, evaluate $\int_0^1 \frac{\log x}{1+x} dx$. Justify the steps made.

4. Let $f_n(x) = \frac{nx}{1+n^2x^2}$, $n \in \mathbb{N}$. Prove or disprove: the sequence (f_n) is uniformly convergent on $[0, 1]$.

5. Prove the following integral form of the Mean Value Theorem: If f and g are continuous on $[a, b]$ and $g(x) \neq 0$ for all $x \in (a, b)$, then there exists $c \in [a, b]$ such that

$$\int_a^b f(x)g(x) dx = f(c) \int_a^b g(x) dx.$$

6. Let $f: [0, 1] \rightarrow \mathbb{R}$ be continuous. Determine $\lim_{n \rightarrow \infty} \int_0^1 nx^n f(x) dx$. Prove your answer.

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ANALYSIS QUALIFYING EXAM – SEPT 24, 2008
Vitaly Bergelson, Ombudsman

1. Find the limit:

$$\lim_{n \rightarrow \infty} n(\sqrt[n]{n} - 1).$$

2. Find the value of the series:

$$\sum_{k=1}^{\infty} \frac{k^2}{2^k}.$$

3. Define $f : \mathbb{R} \rightarrow \mathbb{R}$ by

$$f(x) = \begin{cases} e^{-1/x} & \text{if } x > 0, \\ 0 & \text{if } x \leq 0. \end{cases}$$

Prove that f is infinitely differentiable on \mathbb{R} .

4. Let

$$I = \int_0^{\infty} \frac{\sqrt{x} \cos x}{x + 100} dx.$$

Is I convergent?

5. Suppose that $f_n \in C[0, 1]$ for every n , $f_n(x) \geq f_{n+1}(x)$ for every n and x , and

$$\lim_{n \rightarrow \infty} f_n(x) = f(x)$$

for some function $f \in C[0, 1]$. Show that f_n converges to f uniformly on $[0, 1]$.

6. Let $f : [-1, 1] \rightarrow \mathbb{R}$ be continuous. Prove that

$$\int_{-1}^1 \frac{uf(x)}{u^2 + x^2} dx \rightarrow \pi f(0) \text{ as } u \rightarrow 0^+.$$

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1. For each $n \in \mathbf{N}$, let

$$S_n = \frac{1}{2} + \frac{3}{2^2} + \frac{5}{2^3} + \cdots + \frac{2n-1}{2^n}.$$

Prove that $S_n \rightarrow 3$ as $n \rightarrow \infty$.

2. Let $f: \mathbf{R} \rightarrow \mathbf{R}$ be twice differentiable and suppose $f''(x) > 0$ for all $x \in \mathbf{R}$. Prove that f is strictly convex on \mathbf{R} ; in other words, prove that for all $s, t \in (0, 1)$ with $s + t = 1$ and for all $u, v \in \mathbf{R}$ with $u \neq v$, we have

$$f(su + tv) < sf(u) + tf(v).$$

3. Let (z_n) be a sequence of non-zero complex numbers. Suppose that

$$\limsup_{n \rightarrow \infty} \left| \frac{z_{n+1}}{z_n} \right| < 1.$$

Prove that $\sum_{n=1}^{\infty} z_n$ converges.

4. Prove the following special case of the Riemann-Lebesgue lemma: Let $f: [0, 1] \rightarrow \mathbf{R}$ be continuous. Then

$$\lim_{t \rightarrow \infty} \int_0^1 f(x) \cos(tx) dx = 0.$$

5. Let $p \in (-1, 0]$. Prove that there exists a convergent sequence (α_n) in \mathbf{R} such that for each $n \in \mathbf{N}$, we have

$$\sum_{k=1}^n k^p = \frac{1}{1+p} n^{1+p} + \alpha_n.$$

6. Let $f: [0, 2] \rightarrow \mathbf{R}$ be continuously differentiable. (Use the appropriate one-sided derivatives at 0 and at 2.) For each $n \in \mathbf{N}$, define $g_n: [0, 1] \rightarrow \mathbf{R}$ by $g_n(x) = n(f(x + 1/n) - f(x))$. Prove that the sequence (g_n) converges uniformly on $[0, 1]$.

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1. Let (a_k) be a decreasing sequence of positive real numbers. Suppose that the series $\sum_{m=1}^{\infty} m a_{m^2}$ converges. Prove that the series $\sum_{k=1}^{\infty} a_k$ converges.
2. Let $f(x) = x e^{-x^2} \int_0^x e^{s^2} ds$ for all $x \in \mathbf{R}$. Is f bounded on \mathbf{R} ?
3. Let $f \in C([a, b])$, where $a, b \in \mathbf{R}$ with $a < b$. Let

$$M = \sup \{ |f(x)| : x \in [a, b] \}$$

and for each $n \in \mathbf{N}$, let

$$I_n = \left(\int_a^b |f(x)|^n dx \right)^{1/n}.$$

Prove that $\lim_{n \rightarrow \infty} I_n = M$.

4. Let $f: (0, \infty) \rightarrow \mathbf{R}$ be C^1 . Suppose that f has at least one zero and that $f(x) \rightarrow 0$ as $x \rightarrow \infty$.
 - (a) Prove that f' has at least one zero.
 - (b) Suppose in addition that f is C^2 and that f'' has only finitely many zeros. Prove that $f'(x) \rightarrow 0$ as $x \rightarrow \infty$.
5. Prove or disprove: For each uniformly continuous function $f: [0, \infty) \rightarrow \mathbf{R}$, if the improper Riemann integral $\int_0^{\infty} f(t) dt$ converges, then $\lim_{x \rightarrow \infty} f(x) = 0$.
6. Let \mathcal{F} be the collection of all twice continuously differentiable functions f on \mathbf{R} satisfying $f \geq 0$ on \mathbf{R} and $f'' \leq 1$ on \mathbf{R} . Find a constant $C \in (0, \infty)$ such that for each $f \in \mathcal{F}$ and for each $x \in \mathbf{R}$, we have

$$f'(x)^2 \leq C f(x).$$

Justify the value you find for C .

Analysis Qualifying Examination

Spring, 2007

VITALY BERGELSON, Ombudsman

To pass, it is sufficient to solve four problems completely. You should exercise good judgment in deciding what constitutes an adequate solution. In particular, you should not try to solve a problem by just quoting a theorem that reduces what you are asked to prove to a triviality. If you are not sure whether you may use a particular theorem, ask the proctor.

1. Let $f: [0, \infty) \rightarrow [0, \infty)$ be decreasing and suppose that $\int_0^\infty f(x) dx$ converges. Prove that

$$\lim_{h \rightarrow 0+} \left(h \sum_{n=1}^{\infty} f(nh) \right) = \int_0^\infty f(x) dx.$$

2. Let $C[0, 1]$ denote the set of continuous real-valued functions on $[0, 1]$. Let $p \in (1, \infty)$. For each continuous function $f: [0, 1] \rightarrow \mathbf{R}$, let

$$\|f\| = \left(\int_0^1 |f|^p \right)^{1/p}.$$

Prove Minkowski's inequality in this setting, namely

$$\|f + g\| \leq \|f\| + \|g\|$$

for all $f, g \in C[0, 1]$.

3. Let $f: [0, 1] \rightarrow \mathbf{R}$ be continuous. Define $\varphi: [0, 1] \rightarrow \mathbf{R}$ by

$$\varphi(x) = \int_0^x e^{-xt} f(t) dt.$$

Prove that φ is differentiable and find φ' .

4. Prove that the equation

$$1 - x + \frac{x^2}{2} - \frac{x^3}{3} + \cdots + (-1)^n \frac{x^n}{n} = 0$$

has exactly one solution in \mathbf{R} if n is odd and no solutions if n is even.

5. Let (a_n) be a decreasing sequence of positive numbers such that $\sum_{n=1}^\infty a_n < \infty$. Prove that $na_n \rightarrow 0$ as $n \rightarrow \infty$.

6. Let f be a continuous function on $[0, 1]$. Find

$$\lim_{n \rightarrow \infty} n \int_0^1 x^{n+2} f(x) dx.$$

Justify your answer.

To pass, it is sufficient to solve four problems completely. You should exercise good judgment in deciding what constitutes an adequate solution. In particular, you should not try to solve a problem by just quoting a theorem that reduces what you are asked to prove to a triviality. If you are not sure whether you may use a particular theorem, ask the proctor.

1. Prove the following version of l'Hospital's rule: Let $f, g: \mathbf{R} \rightarrow \mathbf{R}$ be differentiable, with $g'(x)$ never 0, and suppose that as $x \rightarrow \infty$, we have $g(x) \rightarrow \infty$ and $f'(x)/g'(x) \rightarrow \infty$. Then $f(x)/g(x) \rightarrow \infty$ also.
2. Define $f: [0, 1] \rightarrow \mathbf{R}$ by

$$f(x) = \begin{cases} 0 & \text{if } x \in [0, 1] \setminus \mathbf{Q}, \\ \frac{1}{q} & \text{if } x = \frac{p}{q} \in [0, 1] \cap \mathbf{Q}, \end{cases}$$

where p and q are coprime integers. Prove that f is Riemann integrable.

3. Let $a_k \geq 0$ for each $k \in \mathbf{N}$. Prove that if $\sum_{k=1}^{\infty} a_k$ converges, then $\sum_{k=1}^{\infty} \frac{\sqrt{a_k}}{k}$ also converges.
4. Let (f_n) be a sequence of functions from $[a, b]$ to \mathbf{R} , where $a, b \in \mathbf{R}$ with $a < b$. Suppose that for each $c \in [a, b]$, (f_n) is equicontinuous at c and $(f_n(c))$ converges. Prove that (f_n) converges uniformly. (To say that (f_n) is *equicontinuous at c* means that for each $\varepsilon > 0$, there exists $\delta > 0$ such that for each $x \in [a, b]$, if $|x - c| < \delta$, then for each n , $|f_n(x) - f_n(c)| < \varepsilon$.)
5. For each $\alpha \in (0, \infty)$, define $f_\alpha: (0, \infty) \rightarrow \mathbf{R}$ by $f_\alpha(x) = x^\alpha \log x$. For which values of α is f_α uniformly continuous? Justify your answer.
6. Consider the integral

$$I = \int_0^1 \int_0^1 \frac{1}{1 - xy} dx dy.$$

Find $a, b \in \mathbf{R}$ such that $I \in [a, b]$ and $b - a < 1/2$. Justify your answer.

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1. Determine the radius of convergence and behaviour at the endpoints of the interval of convergence for the series

$$\sum_{n=1}^{\infty} \left(1 + \frac{1}{2} + \cdots + \frac{1}{n}\right) x^n.$$

2. Let $n \in \mathbb{N}$. Define $P: \mathbb{R} \rightarrow \mathbb{R}$ by

$$P(x) = \frac{d^n}{dx^n} ((x^2 - 1)^n).$$

Clearly P is a polynomial. Prove that the roots of P are all real and lie in the interval $(-1, 1)$.

3. Let (x_n) be a sequence of strictly positive real numbers. Suppose that $\sum_{n=1}^{\infty} x_n y_n$ converges for each sequence (y_n) of strictly positive real numbers such that $y_n \rightarrow 0$ as $n \rightarrow \infty$. Prove that $\sum_{n=1}^{\infty} x_n < \infty$.

4. Find the limit of $m \sum_{n=m}^{\infty} \frac{1}{n^2}$ as $m \rightarrow \infty$. Justify your answer.

5. Let $n \in \mathbb{N}$ and let $a_1, a_2, \dots, a_n \in (0, \infty)$. Let

$$G = (a_1 a_2 \cdots a_n)^{1/n} \quad \text{and} \quad A = \frac{a_1 + a_2 + \cdots + a_n}{n}.$$

Prove that $G \leq A$.

6. Let $f: [0, \infty) \rightarrow \mathbb{R}$. Suppose that f is uniformly continuous and that

$$\int_0^{\infty} f(x) dx$$

converges. Prove that

$$\lim_{x \rightarrow \infty} f(x) = 0.$$

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1. Determine whether $\sum_{n=1}^{\infty} a_n$ converges, where

$$a_n = \begin{cases} n^{-1} & \text{if } n \text{ is a square,} \\ n^{-2} & \text{otherwise.} \end{cases}$$

Justify your answer.

2. Let $f: \mathbb{R} \rightarrow \mathbb{R}$ be twice continuously differentiable. Suppose that $|f''(x)| \leq 1$ for all x and that $f(-1) = f'(-1) = f(1) = f'(1) = 0$. What is the maximum possible value for $f(0)$? Justify your answer.
3. Prove or disprove: The series $\sum_{n=1}^{\infty} \frac{\sin x}{1 + n^2 x^2}$ converges uniformly on $[-\pi, \pi]$.
4. Suppose that $f: [0, 1] \rightarrow \mathbb{R}$ is continuous and has a local maximum at each point in $[0, 1]$. Prove that f is constant.
5. Prove or disprove: For each unbounded open set $U \subseteq (0, \infty)$, the function f defined by $f(x) = x^2$ is not uniformly continuous on U .
6. Let (a_n) be a sequence of strictly positive real numbers. Prove that

$$\liminf \frac{a_{n+1}}{a_n} \leq \liminf a_n^{1/n}.$$

To pass, it is sufficient to solve four problems completely. You should exercise good judgment in deciding what constitutes an adequate solution. In particular, you should not try to solve a problem by just quoting a theorem that reduces what you are asked to prove to a triviality. If you are not sure whether you may use a particular theorem, ask the proctor.

1. Let

$$x_n = 1 + \frac{1}{2} + \cdots + \frac{1}{n} - \log n$$

for each $n \in \mathbf{N}$. Prove that x_n converges as $n \rightarrow \infty$.

2. Prove or disprove: For each continuous function $f: [0, \infty) \rightarrow \mathbf{R}$, if $\lim_{t \rightarrow \infty} f(t) = \infty$, then

$$\lim_{T \rightarrow \infty} \frac{1}{T} \int_0^T f(t) dt = \infty.$$

3. Prove or disprove: For each function $f: \mathbf{R} \rightarrow \mathbf{R}$ such that f is differentiable at 0, and for each strictly decreasing sequence (a_n) in $(0, \infty)$ such that $\lim_{n \rightarrow \infty} a_n = 0$, we have

$$\lim_{n \rightarrow \infty} \frac{f(a_n) - f(a_{n+1})}{a_n - a_{n+1}} = f'(0).$$

4. Let f be an n times continuously differentiable real-valued function on $[a, b]$, where $a, b \in \mathbf{R}$ with $a < b$. Suppose that the n -th derivative of f satisfies $f^{(n)}(x) > 0$ for each $x \in [a, b]$. Prove that f has at most n zeroes in $[a, b]$.

5. Let $f: [0, 1] \rightarrow \mathbf{R}$ be continuous. Suppose that

$$\int_0^1 f(x)g'(x) dx = 0$$

for each continuously differentiable function $g: [0, 1] \rightarrow \mathbf{R}$ satisfying $g(0) = 0 = g(1)$. Prove that f must be a constant function.

6. Prove that

$$f(x) = \sum_{n=1}^{\infty} (-1)^{n+1} \log\left(1 + \frac{x}{n}\right)$$

is defined and differentiable on the open interval $-1 < x < \infty$.

Analysis Qualifying Examination**Autumn, 2004**

To pass, it is sufficient to solve four problems completely. You should exercise good judgment in deciding what constitutes an adequate solution. In particular, you should not try to solve a problem by just quoting a theorem that reduces what you are asked to prove to triviality. If you are not sure whether you may use a particular theorem, ask a proctor.

Please, begin each problem on a new page. Remember to write your code name at the top of each page. Since the problems will be graded from photocopies, please write only inside the margins that are marked on each page of answer paper and do not write on the backs of the pages of answer paper.

1. Suppose that (a_n) is a decreasing sequence of positive real numbers such that $\sum a_n$ diverges. Prove that $\lim_{n \rightarrow \infty} \frac{a_1 + a_3 + \dots + a_{2n+1}}{a_2 + a_4 + \dots + a_{2n}} = 1$.

2. Suppose that f is a C^1 function on \mathbb{R} which has the properties that $\lim_{x \rightarrow +\infty} f(x) = A$ and $\lim_{x \rightarrow +\infty} f'(x) = B$ for some real numbers A and B . Show that $B = 0$.

3. Let $\varphi \in C_0^\infty(-1, 1)$ (recall that this means that φ is infinitely differentiable and φ is identically 0 in some neighborhood of -1 and 1). Show that for any natural number N , there exists a constant $C = C_N$ such that

$$\left| \int_{-1}^1 e^{i\lambda x} \varphi(x) dx \right| \leq C \lambda^{-N}$$

for all $\lambda > 0$.

4. Let f be a differentiable real valued function on $[1, \infty)$ and suppose that $f'(x)/x$ is bounded. Prove that the function $f(x)/x$ is uniformly continuous on $[1, \infty)$.

5. Let $f(x) = \sum_{n=1}^{\infty} \frac{1}{x + n^{7/5}}$ for $x \geq 0$.

(a) Find $\lim_{x \rightarrow +\infty} f(x)$.

(b) Find $\lim_{x \rightarrow +\infty} \frac{\log f(x)}{\log x}$.

6. If f is a differentiable strictly increasing function on $[0, 1]$, can the set $\{x : f'(x) = 0\}$ be uncountable? (You have to justify your answer, of course.)

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Please begin each problem on a new page. Remember to write your code name at the top of each page. Since the problems may be graded from photocopies, please write only inside the margins that are marked on each page of answer paper and do not write on the backs of the pages of answer paper.

1. Let $f: [-1, 1] \rightarrow \mathbb{R}$ be continuous. Show that

$$\lim_{u \rightarrow 0^+} \int_{-1}^1 \frac{uf(x)}{u^2 + x^2} dx = \pi f(0).$$

2. Let (a_n) be a sequence of real numbers. Suppose that the series $\sum a_n y_n$ converges for every sequence (y_n) with $\lim y_n = 0$. Prove that $\sum a_n$ converges absolutely.
3. Define $h(x) = \sqrt{x^2 + 1}$ for $x \in \mathbb{R}$. Is h uniformly continuous on \mathbb{R} ? Why?
4. Let U be an open set in \mathbb{R} . Show that U may be written as a countable (or finite) disjoint union of open intervals.
5. If a function $g: \mathbb{R} \rightarrow \mathbb{R}$ is differentiable everywhere and g' is one-to-one, prove that g' is monotone.
6. Let the function φ be continuous on $[0, 1]$ with

$$\int_0^1 \varphi(x) dx = 0 \quad \text{and} \quad \int_0^1 x\varphi(x) dx = 1.$$

Prove that $|\varphi(x)| \geq 4$ for some $x \in [0, 1]$.

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Please begin each problem on a new page. Remember to write your code name at the top of each page. Since the problems may be graded from photocopies, please write only inside the margins that are marked on each page of answer paper and do not write on the backs of the pages of answer paper.

1. Prove or disprove: $g(x) = \sin(e^x)$ is uniformly continuous on \mathbb{R} .
2. Suppose (f_n) is a sequence of functions on $[0, 1]$ which converges pointwise to a continuous function f and suppose that for each n , the function f_n is increasing on $[0, 1]$. Does it follow that $f_n \rightarrow f$ uniformly? Justify your answer.
3. Let $a_n \downarrow 0$ with $\sum_{n=1}^{\infty} a_n < \infty$. Prove that $na_n \rightarrow 0$.
4. Construct (or prove the existence of) a continuous function f on $[0, \infty)$ such that the improper integrals $\int_0^{\infty} f(x) dx$ and $\int_0^{\infty} xf(x) dx$ are both well-defined and equal to zero, but $\int_0^{\infty} |f(x)| dx = \infty$.
5. Suppose f' exists and is decreasing on $[0, \infty)$ and $f(0) = 0$. Prove that $\frac{f(x)}{x}$ is decreasing in $(0, \infty)$.
6. Let $f \in C^2[a, b]$, where $a, b \in \mathbb{R}$ with $a < b$. Let $m = (a + b)/2$, the midpoint of the interval $[a, b]$. Prove that there exists $c \in (a, b)$ such that

$$\int_a^b f(x) dx = (b - a)f(m) + \frac{1}{24}f''(c)(b - a)^3$$

Analysis Qualifying Examination

Spring, 2003

To pass, it is sufficient to solve four problems completely. You should exercise good judgment in deciding what constitutes an adequate solution. In particular, you should not try to solve a problem by just quoting a theorem that reduces what you are asked to prove to a triviality. If you are not sure whether you may use a particular theorem, ask the proctor.

Please begin each problem on a new page. Remember to write your code name at the top of each page. Since the problems may be graded from photocopies, please write only inside the margins that are marked on each page of answer paper and do not write on the backs of the pages of answer paper.

1. Find a triangle ABC of maximum area if $A = (-1, 1)$, $B = (2, 4)$, and

$$C \in \{(x, y) : y = x^2, -2 \leq x \leq 2\}.$$

2. Let f_n be differentiable on $[0, 1]$ and suppose:

- (a) For each $n \in \mathbb{N}$ and each $x \in [0, 1]$, $|f'_n(x)| \leq 1$;
- (b) For each $q \in \mathbb{Q} \cap [0, 1]$, the sequence of numbers $(f_n(q))$ converges.

Prove that the sequence of functions (f_n) converges uniformly on $[0, 1]$.

3. Let f be twice differentiable on \mathbb{R} and suppose there are constants $A, C \in [0, \infty)$ such that for each $x \in \mathbb{R}$, $|f(x)| \leq A$ and $|f''(x)| \leq C$. Prove that there is a constant $B \in [0, \infty)$ such that for each $x \in \mathbb{R}$, $|f'(x)| \leq B$.

4. Let f be a continuous function on $[0, 1]$. Determine

$$\lim_{n \rightarrow \infty} n \int_0^1 e^{n(x-1)} f(x) dx.$$

5. Either prove the following statement, or disprove it by giving a counterexample: For each nonnegative continuous function f on $[0, \infty)$, if the improper Riemann integral $\int_0^\infty f dx$ converges, then $\int_0^\infty f^3 dx$ converges.

6. Let K be a compact subset of \mathbb{R}^n and let f be a map from K to K . Consider the graph of f :

$$G_f = \{(x, f(x)) : x \in K\}.$$

Prove that if G_f is a closed subset of $K \times K$, then f is continuous.

To pass, it is sufficient to solve four problems completely. You should exercise good judgment in deciding what constitutes an adequate solution. In particular, you should not try to solve a problem by just quoting a theorem that reduces what you are asked to prove to a triviality. If you are not sure whether you may use a particular theorem, ask the proctor.

Please begin each problem on a new page. Remember to write your code name at the top of each page. Since the problems may be graded from photocopies, please write only inside the margins that are marked on each page of answer paper and do not write on the backs of the pages of answer paper.

1. Determine whether the sequence of functions

$$F_n(x) = \sum_{k=1}^n (-1)^{k-1} \frac{x^{2k-1}}{(2k-1)!}, \quad n = 1, 2, 3, \dots,$$

converges uniformly on the whole real line.

2. Prove that the sequence

$$x_n = 1 + \frac{1}{2} + \frac{1}{3} + \dots + \frac{1}{n} - \ln n, \quad n = 1, 2, 3, \dots,$$

converges as $n \rightarrow \infty$. (Here $\ln n$ means the natural logarithm of n .)

3. Let K be a compact subset of \mathbb{R}^2 and let $f: K \rightarrow \mathbb{R}$ be continuous. Prove that f is uniformly continuous.
4. Let $f: (a, b) \rightarrow \mathbb{R}$ be a convex function. (To say that f is convex means that for all $x_0, x_1 \in (a, b)$ and all $t \in [0, 1]$, we have $f((1-t)x_0 + tx_1) \leq (1-t)f(x_0) + tf(x_1)$.) Prove that the right hand derivative of f exists and is finite at every point of (a, b) . (Of course the same is true for the left hand derivative, although you are not asked to prove this.)
5. Let $f: [0, \infty) \rightarrow \mathbb{R}$ be uniformly continuous and suppose that the improper Riemann integral $\int_0^\infty f(x) dx$ converges. Prove that $f(x) \rightarrow 0$ as $x \rightarrow \infty$.
6. Let $f: \mathbb{R} \rightarrow \mathbb{R}$ be increasing. Prove that f has at most a countable number of discontinuities.

Analysis Qualifying Examination

Spring, 2002

To pass, it is sufficient to solve four problems completely.

Please begin each problem on a new page. Remember to write your code name at the top of each page. Since the problems may be graded from photocopies, please write on only one side of each page and leave adequate margins.

1. Prove this form of Dini's theorem: Let (f_n) be a sequence of ^{continuous!} real-valued functions on the closed bounded interval $[a, b]$. Suppose that for each $t \in [a, b]$, we have $f_n(t) \geq f_{n+1}(t)$ for all n and $\lim_{n \rightarrow \infty} f_n(t) = 0$. Prove that (f_n) converges uniformly to 0 on $[a, b]$.

2. Let f be a differentiable function from \mathbb{R} to \mathbb{R} . Suppose that for each $x \in \mathbb{R}$, we have

$$0 \leq f(x) \leq \frac{1}{1+x^2}.$$

Show that there exists $c \in \mathbb{R}$ such that

$$f'(c) = \frac{-2c}{(1+c^2)^2}.$$

3. Define a function f on the interval $(0, 1)$ by

$$f(x) = \sum_{k=1}^{\infty} \frac{\sin(kx)}{k^2}.$$

Prove that f is differentiable on $(0, 1)$.

4. Let f be a continuously differentiable function from the interval $[0, 1]$ to \mathbb{R} . (Use one-sided derivatives at the endpoints of the interval.) Suppose that $f(1/2) = 0$. Show that

$$\int_0^1 |f(x)|^2 dx \leq \int_0^1 |f'(x)|^2 dx.$$

5. Let f be a continuous function from the interval $[0, 1]$ to \mathbb{R} . Compute

$$\lim_{n \rightarrow \infty} n \int_0^1 x^n f(x) dx.$$

Justify your answer.

6. Let (x_n) be a sequence of real numbers. For each n , let

$$A_n = \frac{x_1 + \cdots + x_n}{n}.$$

Suppose $x_n \rightarrow \infty$ as $n \rightarrow \infty$. Show that $A_n \rightarrow \infty$ as $n \rightarrow \infty$.

Qualifying Exam in Analysis

October 1, 2001

Four complete solutions are sufficient for passing.

Please, don't write on the back of sheets. Use distinct sheets for distinct problems. Put your code name and problem number at the top of each sheet. Number separately the pages related to distinct problems.

Attention: If you use a theorem, please formulate it and verify its applicability.

1. Let R be a non-constant rational function, $R(x) = \frac{a_m x^m + \dots + a_1 x + a_0}{b_n x^n + \dots + b_1 x + b_0}$, $a_m, b_n \neq 0$. Prove that there exists $x_0 \in \mathbb{R}$ such that R is strictly monotone on (x_0, ∞) .

2. Find (i) $\lim_{n \rightarrow \infty} \sin(\pi\sqrt{n^2 + 1})$;
(ii) $\lim_{n \rightarrow \infty} \sin^2(\pi\sqrt{n^2 + n})$.

3. Let f be a continuous function on $[0, 1]$. Prove that

$$\exp\left(\int_0^1 f(x) dx\right) \leq \int_0^1 \exp(f(x)) dx.$$

4. (i) Prove that for every $n = 1, 2, \dots$ the series $\sum_{k=n+1}^{\infty} \left(\frac{1}{\sqrt{k-n}} - \frac{1}{\sqrt{k}}\right)$ converges.

(ii) For each $n \in \mathbb{N}$, let S_n be the sum of the above series. Evaluate $\lim_{n \rightarrow \infty} \frac{S_n}{\sqrt{n}}$.

5. Prove the theorem on term-by-term differentiation of power series: if $r > 0$ and $f(x) = \sum_{n=0}^{\infty} a_n x^n$ converges for $-r < x < r$, then $g(x) = \sum_{n=1}^{\infty} n a_n x^{n-1}$ also converges for $-r < x < r$, and $f' = g$.

6. Let a function f be uniformly continuous on $[1, \infty)$. Prove that the function $F(x) = \frac{f(x)}{x}$ is bounded on $[1, \infty)$.

Qualifying Exam in Analysis

March 26, 2001

Four complete solutions are sufficient for passing.

Please, use only one side of every sheet, and use distinct sheets for distinct problems. Write your code name at the top of each sheet.

1. State and prove Cauchy's inequality for real sequences $(a_k)_{k=1}^{\infty}$ and $(b_k)_{k=1}^{\infty}$, and obtain necessary and sufficient conditions when it is an equality.

2. How many positive solutions does the equation $e^x = 4\cos x$ have? Prove your answer.

3. If f is a convex functions on $[0, 1]$, prove that $f(x) + f(1 - x)$ is decreasing on $[0, \frac{1}{2}]$.

(A function f is *convex* on $[a, b]$ if for any $x, y \in [a, b]$ and any $0 \leq t \leq 1$ it satisfies $f(tx + (1 - t)y) \leq tf(x) + (1 - t)f(y)$. Note that f is not assumed to be differentiable.)

4. Let $a > 1$. Find $\lim_{n \rightarrow \infty} (a - \sqrt[n]{a})^n$.

5. Consider the series

$$x^2 + \frac{x^2}{1+x^2} + \frac{x^2}{(1+x^2)^2} + \dots + \frac{x^2}{(1+x^2)^n} + \dots$$

(i) Prove that the series converges for every real x .

(ii) Does the series converge uniformly on \mathbb{R} ? Prove your answer.

6. Let $f: [0, \infty) \rightarrow \mathbb{R}$ be non-negative and continuous and let $\int_0^{\infty} f(t) dt$ converge.

(i) Show that these conditions do not imply that $f(x) \rightarrow 0$ as $x \rightarrow \infty$.

(ii) Show that under the additional condition that f is uniformly continuous on $[0, \infty)$, $f(x) \rightarrow 0$ as $x \rightarrow \infty$.

AU 2000 ANALYSIS QUALIFYING EXAMINATION

Four complete solutions are sufficient for passing.

Please, use only one side of every sheet, and use distinct sheets for distinct problems!

1. Let $a_n > 0$ for all $n \geq 1$. Suppose that there exists

$$q = \lim_{n \rightarrow \infty} \frac{a_{n+1}}{a_n}.$$

Prove that $\lim_{n \rightarrow \infty} a_n^{\frac{1}{n}}$ exists too, and

$$\lim_{n \rightarrow \infty} a_n^{\frac{1}{n}} = q.$$

2. Determine all real numbers α such that the sequence $g_n(x) = x^\alpha e^{-nx}$ is uniformly convergent on $(0, \infty)$ as $n \rightarrow \infty$.

3. Prove that the theta-function

$$\theta(x) = \sum_{n=-\infty}^{\infty} e^{-\pi n^2 x}$$

is well defined and infinitely differentiable for $x > 0$.

4. Suppose $1 < \alpha < 1 + \beta$. Prove that the function

$$f(x) = \begin{cases} x^\alpha \sin \frac{1}{x^\beta}, & \text{if } x \neq 0, \\ 0, & \text{if } x = 0 \end{cases}$$

is differentiable on $[0, 1]$, but its derivative is unbounded on $[0, 1]$.

5. Let $f: [0, 1] \rightarrow [0, 1]$ be increasing. Using the partition definition of the Riemann integral, prove that f is Riemann integrable.

6. Suppose that $f: [0, \infty) \rightarrow \mathbb{R}$ is continuous. Prove that if

$$\int_0^\infty f(x) dx = \lim_{A \rightarrow \infty} \int_0^A f(x) dx$$

exists (in short, $\int_0^\infty f(x) dx$ converges), then $\int_0^\infty e^{-\alpha x} f(x) dx$ converges for every $\alpha > 0$ and

$$\lim_{\alpha \rightarrow 0^+} \int_0^\infty e^{-\alpha x} f(x) dx = \int_0^\infty f(x) dx.$$

Analysis Qualifying Examination March 27, 2000

Solve the problems on the sheets provided. Use your code name. Work separate problems on separate sheets. Write on one side only, and keep adequate margins. Four problems, done correctly and completely, will be sufficient to pass.

1. Determine whether or not the following statement is correct. If it is correct, prove it. If it is not, provide a counterexample.

Let $f: [0, \infty) \rightarrow \mathbb{R}$ be continuous, and let $\lim_{x \rightarrow \infty} f(x) = 0$.

Then f is uniformly continuous on $[0, \infty)$.

2. Determine all real numbers α for which the improper integral

$$\int_1^{\infty} x^{\alpha} \sin(x^2) dx \quad \text{(i) converges absolutely, (ii) converges.}$$

3. Let (a_i) be a real sequence and let a be a real number. We say that a_i converges to a in the sense of Cesàro if and only if

$$\lim_{n \rightarrow \infty} (1/n) \sum_{i=1}^n a_i = a.$$

Show that if a_i converges to a in the usual sense, then a_i also converges to a in the sense of Cesàro.

4. Let $a > 0$ and $\epsilon > 0$ be given. Find a positive constant K , depending on a and ϵ , such that for all $x > K$, the inequality $\log x < \epsilon x^a$ holds. Verify that your constant works. It is not necessary to find the least possible K .

5. For a bounded interval $[a, b]$, let M be the set of all continuous strictly positive functions on $[a, b]$. For f in M , let $L(f)$ be defined by

$$L(f) := \left\{ \int_a^b f(x) dx \right\} \left\{ \int_a^b (1/f(x)) dx \right\}.$$

For what function(s) f does $L(f)$ attain its minimum value, and what is that minimum value?

6. Prove the following (Arithmetic-Geometric Mean) inequality. If n is a positive integer, and if x_1, \dots, x_n are positive numbers, then $(x_1 x_2 \dots x_n)^{1/n} \leq (1/n) (x_1 + x_2 + \dots + x_n)$

AU 1999 ANALYSIS QUALIFYING EXAMINATION

1. (20 pts.) Prove that if $(s_n)_{n \in \mathbb{N}}$ is a sequence of real numbers, and if the finite $\lim_{n \rightarrow \infty} s_n$ exists, then

$$\sum_{n=0}^{\infty} s_n x^n$$

converges for all x in $(-1, 1)$, and, furthermore,

$$\lim_{x \rightarrow 1^-} (1-x) \sum_{n=0}^{\infty} s_n x^n = \lim_{n \rightarrow \infty} s_n.$$

2. (20 pts.) Give an example of two sequences of real numbers $(a_n)_{n \in \mathbb{N}}$ and $(b_n)_{n \in \mathbb{N}}$ such that $\sum_{n=0}^{\infty} a_n$ converges, $\lim_{n \rightarrow \infty} b_n = 0$, $(b_n)_{n \in \mathbb{N}}$ is positive, and $\sum_{n=0}^{\infty} a_n b_n$ diverges.

3. (20 pts.) Let f be a real valued function such that $f \in C([0, \infty))$, and the improper Riemann integral $\int_1^{\infty} f(x)/x \, dx$ converges. Prove that for all $a > 0$ and $b > 0$ we have

$$\lim_{\epsilon \rightarrow 0^+} \int_{\epsilon}^{\infty} \frac{f(ax) - f(bx)}{x} \, dx = f(0) \log \frac{b}{a}.$$

4. (20 pts.) Prove that

$$\left(\frac{n}{n+1} \right)^n, \quad n \in \mathbb{N},$$

is a decreasing sequence.

5. (20 pts.) Let $f : \mathbb{R} \rightarrow \mathbb{R}$ be differentiable in \mathbb{R} . Suppose $f'(-1) = -1$ and $f'(1) = 1$. Prove that f' must vanish at some point in $(-1, 1)$ [do not use Darboux's Theorem].

6. (20 pts.) Let $(f_n)_{n \in \mathbb{N}}$ be defined by

$$f_n(x) \stackrel{\text{def}}{=} \frac{n^2 x^2}{1 + n^2 x^2}, \quad -1 \leq x \leq 1.$$

- (i) (5 pts.) Find $f \stackrel{\text{def}}{=} \lim_{n \rightarrow \infty} f_n$ in $[-1, 1]$.
 (ii) (5 pts.) Prove or disprove that

$$\lim_{n \rightarrow \infty} \int_{-1}^1 f_n(t) \, dt = \int_{-1}^1 f(t) \, dt.$$

- (iii) (10 pts.) Prove or disprove that $\lim_{n \rightarrow \infty} f_n = f$ is uniform in $[-1, 1]$.

SPRING 1999 ANALYSIS QUALIFYING EXAM

1. Find a sequence of functions $\{f_n\} \in C([0, 1])$ such that $f_n \rightarrow f$ ($n \rightarrow \infty$) with $f \in C([0, 1])$, but

$$\int_0^1 f_n \not\rightarrow \int_0^1 f, \quad n \rightarrow \infty.$$

2. Let $a_n \in \mathbb{R}^+$, $n \in \mathbb{N}$. Prove that if $\sum_{j=1}^{\infty} a_n$ converges, then

$$\sum_{j=1}^{\infty} \frac{\sqrt{a_n}}{n}$$

converges as well.

3. Let f be a C^2 function on the real line. For $k = 0, 1, 2$, define

$$M_k = \sup_{x \in \mathbb{R}} |f^{(k)}(x)|,$$

where $f^{(k)}$ denotes the k -th derivative of f . Prove that there exists an absolute constant K [independent of the function f and its derivatives] such that

$$M_1 \leq K \sqrt{M_0 \cdot M_2}.$$

Hint: you may wish to use Taylor's theorem.

4. For $n \in \mathbb{N}$ and $x \in \mathbb{R}^+$, define

$$f_1(x) = \sqrt{x}$$

$$f_{n+1}(x) = \sqrt{x + f_n(x)}.$$

Prove that $\{f_n\}$ converges uniformly on every interval $[a, b]$ where $0 < a < b < \infty$.

5. Let $f \in C(\mathbb{R})$ and suppose that $\lim_{|x| \rightarrow \infty} f(x) = 0$. Show that f is uniformly continuous on the entire real line.

6. A theorem of Weierstrass says that the uniform limits of polynomials on $[0, 1]$ is the set of all continuous functions on $[0, 1]$. What is the set of uniform limits of polynomials on $[0, 1]$ with only even exponents without constant terms, i.e., polynomials of the form

$$a_2x^2 + a_4x^4 + \cdots + a_{2n}x^{2n}?$$

Four completely solved problems are sufficient for passing. Please write on only one side of each sheet of paper, but put your code on every sheet.

1. For which real x and positive integers n is it true that

$$(1+x)^{2n} \geq 1+2nx \quad ?$$

2. Prove this version of the "root test": Let a_n ($n = 0, 1, 2, \dots$) be real numbers, define R by

$$\frac{1}{R} = \limsup_{n \rightarrow \infty} |a_n|^{1/n}, \quad \text{assume } 0 < R < \infty,$$

and let x be a real number such that $|x| < R$. Show that the series $\sum_{n=0}^{\infty} a_n x^n$ converges.

3. Given $f: [a, b] \rightarrow \mathbb{R}$. Let $a < x < b$ and suppose that there are sequences $\{\alpha_n\}_{n=1}^{\infty}$ and $\{\beta_n\}_{n=1}^{\infty}$ where $a < \alpha_n < x < \beta_n < b$ such that $\alpha_n \rightarrow x$, $\beta_n \rightarrow x$, and the limit

$$\lim_{n \rightarrow \infty} \frac{f(\beta_n) - f(\alpha_n)}{\beta_n - \alpha_n}$$

does not exist. Prove that f is not differentiable at x .

4. Show that if f is Riemann-integrable on $[a, b]$, then the function $g: \mathbb{R} \rightarrow \mathbb{R}$ defined by

$$g(x) = \int_a^b f(t) \sin(xt) dt$$

is uniformly continuous on \mathbb{R} .

5. Determine the subsets of $[0, 2\pi]$ on which the series

$$\sum_{k=1}^{\infty} \frac{\cos(kx)}{\sqrt{k}}$$

is uniformly convergent.

6. Describe all functions $g: [0, \infty) \rightarrow \mathbb{R}$ with the following properties:

- (i) g is continuous on $[0, \infty)$.
- (ii) $\lim_{x \rightarrow \infty} g(x) = 0$.
- (iii) $\int_1^{\infty} g(x) x^k dx = 0$ for $k = -2, -3, -4, \dots$.

7. Let C be a right circular cylinder with base radius 2 cm and height 3 cm. What is the largest area of a plane section of C , assuming that the plane defining the section does not intersect the open discs at the ends of the cylinder?

Analysis Qualifying Exam - September 24, 1997

Use separate sheets of paper for each problem and please write your chosen code name on each sheet. Four complete solutions are sufficient to pass this examination. Good luck!

1.) Prove that $n! \leq \left(\frac{n+1}{2}\right)^n$, for all integers $n \geq 1$.

2.) Prove that

$$\sup_{-\infty < a < b < \infty} \left| \int_a^b \frac{\sin x}{x} dx \right| < \infty$$

3.) Let $f : \mathbb{R} \rightarrow \mathbb{R}$ be a differentiable function such that $\lim_{x \rightarrow \infty} f'(x) = 0$. Show that

$$\lim_{x \rightarrow \infty} \frac{f(x)}{x} = 0$$

(You may not quote l'Hopital's rule without proving it.)

4.) Let $g : \mathbb{R} \rightarrow \mathbb{R}$ be a bounded, continuous function. For each $x \in \mathbb{R}$ compute the limit

$$\lim_{\lambda \rightarrow \infty} \lambda \int_{-\infty}^{\infty} g(x+y) e^{-\lambda|y|} dy$$

5.) Find the minimal area among triangles having vertices $A = (0, -2)$, $B = (4, 0)$ and third vertex C on the parabola $y = x^2$.

6.) The Taylor series expansion of $f(x) = \log(1+x)$ about the point $x = 0$ is the following:

$$\sum_{n=1}^{\infty} (-1)^{n-1} \frac{x^n}{n}$$

Show that the series diverges at every point of $(-\infty, -1] \cup (1, \infty)$ and converges uniformly on every interval of the form $[-1 + \epsilon, 1]$, $0 < \epsilon < 1$.

Spring 1997 #6

Carroll \Rightarrow Edgar \Rightarrow Neri

ANALYSIS QUALIFYING EXAM

Problem 1. Prove that the series

$$\sum_{n=100}^{\infty} \frac{1}{(\log n)^{\log \log n}}$$

diverges.

Problem 2. Let $\{a_n\}_{n=1}^{\infty}$ be a sequence of positive numbers, and let $L \geq 0$. Consider the two properties:

(i)
$$\lim_{n \rightarrow \infty} \frac{a_{n+1}}{a_n} = L$$

and

(ii)
$$\lim_{n \rightarrow \infty} (a_n)^{\frac{1}{n}} = L.$$

Prove that one of these properties implies the other and give an example showing that they are not equivalent.

Problem 3.

Let $f : [-1, 1] \rightarrow \mathbb{R}$ be continuous in $[-1, 1]$, and let it be differentiable in $(-1, 1)$. Prove that

$$\lim_{\epsilon \rightarrow 0} \int_{\epsilon < |t| < 1} \frac{f(t)}{t} dt$$

exists and is finite.

Problem 4. Let $S_n \stackrel{\text{def}}{=} n \sum_{k=n+1}^{2n} k^{-2}$. Prove that $\lim_{n \rightarrow \infty} S_n = \frac{1}{2}$.

Problem 5. State (an appropriate version of) Hölder's inequality, and use it to show that if $f : [0, 1] \rightarrow \mathbb{C}$ satisfies $\int_0^1 |f|^8 < \infty$, then $\int_0^1 |f|^3 < \infty$ as well. In addition, give an example of a function f defined on the whole real line ($f : \mathbb{R} \rightarrow \mathbb{C}$) such that although $\int_{\mathbb{R}} |f|^8 < \infty$, we still have $\int_{\mathbb{R}} |f|^3 = \infty$.

Problem 6. Prove that the sequence $\frac{x^2 + 3x^{2n+4}}{1 + 3x^{2n}}$ converges uniformly in \mathbb{R} .

Av 96

Analysis Qualifying Examination

September 25, 1996

To pass, it is sufficient to solve four problems completely.

\mathbb{R} denotes the set of real numbers and \mathbb{N} denotes the set $\{1, 2, 3, \dots\}$ of natural numbers.

1. Let f be a positive continuous function on $[0, \infty)$. Assume that

$$\lim_{x \rightarrow \infty} \frac{f(x+1)}{f(x)}$$

exists and is equal to $L < 1$. Prove that the improper integral $\int_0^\infty f(x) dx$ converges.

2. Let $a_n > 0$ for all $n \in \mathbb{N}$. Prove that if the series $\sum a_n$ converges, then there exists a sequence $\lambda_n \uparrow \infty$ such that the series $\sum \lambda_n a_n$ converges.

3. Let f be defined and twice continuously differentiable on \mathbb{R} . Suppose that for all $s, t \in \mathbb{R}$ with $s < t$, we have

$$\frac{1}{t-s} \int_s^t f(x) dx = \frac{f(s) + f(t)}{2}.$$

Prove that there exist constants α and β such that for all $x \in \mathbb{R}$, we have

$$f(x) = \alpha x + \beta.$$

4. Find $\lim_{x \rightarrow \infty} x e^{x^2} \int_x^\infty e^{-u^2} du$.

5. (a) For all $n \in \mathbb{N}$, let

$$s_n = 1 + \frac{1}{2} + \dots + \frac{1}{n} - \log n.$$

Prove that the sequence (s_n) converges.

- (b) Find $\lim_{n \rightarrow \infty} \sum_{k=n}^{2n} \frac{1}{k}$.

- (c) Find $\lim_{n \rightarrow \infty} \sum_{k=n}^{2n} \sin \frac{\pi}{k}$.

6. Let (f_n) be a pointwise convergent sequence of continuous real-valued functions on the interval $[0, 1]$. Consider the equation

$$\lim_{n \rightarrow \infty} \lim_{x \rightarrow 0} f_n(x) = \lim_{x \rightarrow 0} \lim_{n \rightarrow \infty} f_n(x). \quad (*)$$

- (a) Show by an example that $(*)$ does not hold in general.

- (b) Prove that if the sequence (f_n) converges uniformly, then $(*)$ holds.

Analysis Qualifying Examination

March 27, 1996

To pass, it is sufficient to solve four problems completely.

1. Let $x_1 = 1$ and for $n = 1, 2, 3, \dots$, let

$$x_{n+1} = \frac{1}{2} \left(x_n + \frac{2}{x_n} \right).$$

Prove that the sequence (x_n) converges and find its limit.

2. Find all values of $a > 0$ for which the series

$$\sum_{n=1}^{\infty} (e^{1/n^a} - 1)^3$$

converges. Justify your answer.

3. For $n = 2, 3, 4, \dots$, let

$$R_n = \frac{1}{\log n} \sum_{k=1}^n \frac{1}{k} \quad \text{and} \quad D_n = \left(\sum_{k=1}^n \frac{1}{k} \right) - \log n.$$

(a) Prove that $R_n \rightarrow 1$ as $n \rightarrow \infty$.

(b) Prove that D_n converges as $n \rightarrow \infty$.

4. Let f be a bounded uniformly continuous function on \mathbb{R} . For $n = 1, 2, 3, \dots$ and for $x \in \mathbb{R}$, let $g_n(x) = ne^{-\pi n^2 x^2}$ and let $f_n(x) = \int_{-\infty}^{\infty} f(x-y)g_n(y) dy$. Prove that $f_n \rightarrow f$ uniformly on \mathbb{R} as $n \rightarrow \infty$. (You may use without proof the fact that $\int_{-\infty}^{\infty} e^{-\pi x^2} dx = 1$.)

5. Let

$$f(t) = \int_0^{\infty} \frac{\cos x}{x+t} dx$$

for all $t > 0$ for which the integral converges. Prove that f is defined and continuous on the interval $0 < t < \infty$.

6. Let x_1, x_2, x_3, \dots be distinct numbers in $[0, 1]$. Let (c_i) be a sequence of real numbers converging to 0. For $t \in [0, 1]$, let

$$f(t) = \begin{cases} c_i & \text{if } t = x_i, \\ 0 & \text{otherwise.} \end{cases}$$

Prove that for each $\epsilon > 0$, there exists $\delta > 0$, such that for each partition

$$P: 0 = t_0 < t_1 < \dots < t_n = 1$$

of the interval $[0, 1]$ whose norm

$$\|P\| = \max \{t_k - t_{k-1} : k = 1, \dots, n\}$$

satisfies $\|P\| < \delta$, and each Riemann sum S , for the function f , corresponding to the partition P , one has $|S| < \epsilon$.

95

Four complete solutions are sufficient for passing.

1. Let $s_1 = 0$ and $s_{n+1} = e^{-s_n}$, $n \geq 1$. Prove that there exists a finite $\lim_{n \rightarrow \infty} s_n$.

2. Let $F : [0, 1] \rightarrow \mathbb{R}$ and $G : [0, 1] \rightarrow \mathbb{R}$ be functions continuous on $[0, 1]$. Prove:

2a. If F, G are differentiable on $(0, 1)$ then there exists a point $c \in (0, 1)$ such that

$$[F(1) - F(0)]G'(c) = [G(1) - G(0)]F'(c).$$

2b. If F, G are twice differentiable on $(0, 1)$ then there exists a point $d \in (0, 1)$ such that

$$\left[F(1) - 2F\left(\frac{1}{2}\right) + F(0)\right]G''(d) = \left[G(1) - 2G\left(\frac{1}{2}\right) + G(0)\right]F''(d).$$

(Note: If you need Rolle's Theorem, you can use it without a proof.)

3. Let $F : [0, \infty) \rightarrow \mathbb{R}$ be given by

$$F(t) = \begin{cases} t \log t, & t > 0, \\ 0, & t = 0. \end{cases}$$

Prove that F is continuous and "approximately additive". (The latter means that there is a constant $c \geq 0$ such that

$$|F(s) + F(t) - F(s+t)| \leq c(s+t), \quad \forall s, t \geq 0.)$$

4. Determine

$$\lim_{t \rightarrow 0} t \cdot \sum_{n=1}^{\infty} \frac{1}{n^2 + t^2}, \quad \lim_{t \rightarrow \infty} t \cdot \sum_{n=1}^{\infty} \frac{1}{n^2 + t^2}.$$

5. Suppose that $f : [0, 1] \rightarrow [0, \infty)$ is continuous, and concave. (The latter means that, for all $x_1, x_2 \in [0, 1]$,

$$f(\lambda_1 x_1 + \lambda_2 x_2) \geq \lambda_1 f(x_1) + \lambda_2 f(x_2), \quad \forall \lambda_1, \lambda_2 \geq 0, \lambda_1 + \lambda_2 = 1.)$$

Show that

$$\int_0^1 x f(x) dx \leq \frac{2}{3} \int_0^1 f(x) dx.$$

For which f does the equality hold in the above expression?

6. Suppose that $f : [0, \infty) \rightarrow \mathbb{R}$ is continuous, and that $\lim_{x \rightarrow \infty} f(x) = 1$. Prove that the improper integral $\int_0^{\infty} e^{-tx} f(x) dx$ converges for every $t > 0$, and that

$$\lim_{t \rightarrow 0^+} t \int_0^{\infty} e^{-tx} f(x) dx = 1.$$

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March 29, 1995
SR

REAL ANALYSIS QUALIFYING EXAM.

Four complete solutions are sufficient for passing this examination. Good luck!

1. Let f and g be continuous functions on $[0, 1]$. Suppose we know that

$$\int_0^1 f(t)t^n dt = \int_0^1 g(t)t^n dt$$

for $n = 1, 2, 3, \dots$ (Notice that the case $n = 0$ is missing.) Can we conclude that $f(t) = g(t)$ everywhere on $[0, 1]$?

2. (a) Prove that

$$(*) \quad x - 2x^2 \leq \log(1+x) \leq x$$

for $x \geq -1/2$. (b) Use (*) to prove the following "absolute convergence test" for infinite products: Suppose $a_n > -1$ for all $n \in \mathbb{N}$ and $\sum_{n=1}^{\infty} |a_n| < \infty$. Then the limit

$$\lim_{M \rightarrow \infty} \prod_{n=1}^M (1 + a_n).$$

exists, and is non-zero.

3. Let X be a compact metric space with metric ρ , and let $f : X \rightarrow \mathbb{R}$ be a continuous function from X into \mathbb{R} which has the following property: For every point $x \in X$ there is another point $x' \in X$ such that

$$|f(x')| \leq \frac{1}{2}|f(x)|.$$

Prove that there exists a point $x_0 \in X$ where f vanishes, $f(x_0) = 0$ that is:

4. Let $\sum_{n=1}^{\infty} a_n$ be a convergent series of nonnegative terms. Prove:

- (a) $\liminf_{n \rightarrow \infty} na_n = 0$;
(b) possibly $\limsup_{n \rightarrow \infty} na_n > 0$; [give an example];
(c) if $\{a_n\}$ is decreasing then $\lim_{n \rightarrow \infty} na_n = 0$.

5. Let $f : [0, T] \rightarrow \mathbb{R}$ be twice differentiable on $[0, T]$, and such that

$$f(T) - f(0) = 1, \quad f'_+(0) = f'_-(T) = 0, \quad |f''(t)| \leq 1, \quad \forall t \in [0, T].$$

Prove that

$$T \geq 2.$$

6. Let g be a continuous function on \mathbb{R} with $g(0) = 0$ and let g' be bounded on \mathbb{R} , that is

$$\sup \{|g'(x)| : x \in \mathbb{R}\} = M < \infty.$$

(a) Show that the series

$$\sum_{n=1}^{\infty} \frac{1}{n} g\left(\frac{x}{n}\right)$$

converges for all $x \in \mathbb{R}$ and that its sum

$$f(x) = \sum_{n=1}^{\infty} \frac{1}{n} g\left(\frac{x}{n}\right)$$

is a continuous function on \mathbb{R} . (b) Is f a differentiable function on \mathbb{R} ?

pg 2
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done

QUALIFYING EXAM IN ANALYSIS

September 21, 1994
Amal

Four complete solutions are sufficient for passing.

1. Prove the Cauchy inequality: Let $x_1, x_2, \dots, x_n, y_1, y_2, \dots, y_n$ be real numbers. Then

$$\left(\sum_{j=1}^n x_j y_j \right)^2 \leq \left(\sum_{j=1}^n x_j^2 \right) \left(\sum_{j=1}^n y_j^2 \right)$$

2. Prove that if $\sum a_n$ is convergent but not absolutely convergent series with real terms, and if c is any real number, then there exists a rearrangement $\sum b_n$ of the series $\sum a_n$ such that $\sum b_n = c$.
3. Let $x_n = \frac{1}{n}$ if n is odd, $x_n = 1 - \frac{1}{n}$ if n is even. For $x \in (0, 1]$ let $f(x) = \sum_{x_n < x} \frac{1}{n^2}$. Describe f , in particular, determine all its discontinuities. Prove your assertions.
4. Let the function f_0 be continuous on $[0, 1]$, let $f_n(x) = \int_0^x f_{n-1}(y) dy$ for $n = 1, 2, \dots$, and let $M_n = \sup_{0 \leq x \leq 1} |f_n(x)|$.
 - (i) Prove that $|f_n(x)| \leq M_0 \frac{x^n}{n!}$ for $x \in [0, 1]$, $n = 1, 2, \dots$.
 - (ii) Deduce that, for any $A > 1$, $A^n M_n \rightarrow 0$, $n \rightarrow \infty$.
5. Prove or disprove.
 - (i) If f is continuously differentiable on $[0, 1]$ then there exists a sequence of polynomials, $\{p_n\}$, such that $p_n \rightarrow f$ uniformly on $[0, 1]$ and $p'_n \rightarrow f'$ uniformly on $[0, 1]$.
 - (ii) If f is infinitely differentiable on $[0, 1]$, then there exists a sequence of polynomials, $\{g_n\}$, such that for every k , $k = 0, 1, 2, \dots$, $g_n^{(k)} \rightarrow f^{(k)}$ as $n \rightarrow \infty$, uniformly on $[0, 1]$.
6. Let f and g be non-negative functions on $[0, \infty)$, both locally Riemann-integrable. Prove that if $\liminf_{x \rightarrow \infty} f(x) \neq 0$, then

$$\int_0^\infty |f(x) \cos x + g(x) \sin x| dx = \infty.$$

ANALYSIS QUALIFYING EXAMINATION

6 p.m. to 9 p.m. EST in MA 050 on Wednesday, March 30, 1994

This is a "CLOSED BOOK" test

Please use a separate sheet of paper for each problem. Write your chosen code name on each piece. Please make the solutions as detailed, as rigorous, as readable, as pedantic, and as legible as possible. If think you must make a reference to a (well known) result then such a reference must be as accurate as possible. You have 180 minutes for solving the problems. Generally speaking, four *complete* solutions are sufficient for passing this examination. Good luck ...

Problem 1. Let $\{a_n \in \mathbb{C}\}_{n \in \mathbb{N}}$ satisfy $\lim_{n \rightarrow \infty} a_n = \alpha$. Prove that

$$\lim_{n \rightarrow \infty} \frac{1}{\log n} \sum_{k=1}^n \frac{a_k}{k}$$

exists and is equal to α .

Problem 2. Let $\{f_n : \mathbb{R} \mapsto [0, 1]\}_{n \in \mathbb{N}}$ be a sequence of nondecreasing continuous functions. Prove that $\{f_n\}_{n \in \mathbb{N}}$ contains a subsequence which converges everywhere.

Problem 3. Let $f \in C([0, 1], \mathbb{R})$. Prove that

$$\lim_{n \rightarrow \infty} \frac{1}{n} \log \left(\int_0^1 \exp(n \cdot f(x)) dx \right) = \sup_{x \in [0, 1]} f(x).$$

Problem 4. Let $\{f_n : \mathbb{R}^+ \mapsto \mathbb{R}^+\}_{n \in \mathbb{N}}$ be defined by $f_n(x) = \frac{x\sqrt{n}}{1+n x^2}$. Prove or disprove (i) that $\{f_n\}_{n \in \mathbb{N}}$ converges uniformly on $[0, 1]$, and that (ii) $\{f_n\}_{n \in \mathbb{N}}$ converges uniformly on $(1, \infty)$.

Problem 5. Let $f : \mathbb{R} \rightarrow \mathbb{R}$ be differentiable in \mathbb{R} . Suppose $f'(0) = -1$ and $f'(1) = 1$. Prove the special case of Darboux's Theorem that f' must vanish at some point in $(0, 1)$ (without using Darboux's Theorem itself).

Problem 6. Prove that $\sum_{k=0}^{\infty} \frac{\sin kx}{k+1}$ converges uniformly for $x \in [\epsilon, \pi/2]$ for all $\epsilon \in (0, \pi/2)$ but the uniform convergence fails on $[0, \pi/2]$.

Qualifying Exam in Analysis

15
Aug 93
September 22, 1999

Four completely solved problems are sufficient for passing the exam.

1. Prove: If $f: [a, b] \rightarrow \mathbb{R}$ is monotone increasing, then the set of discontinuities for f is at most countable.

2. a) Let $x_1 = 1$ and let $x_{n+1} = \frac{1}{x_n + 1}$, $n=1, 2, \dots$

Determine whether $\{x_n\}$ converges, and if so, what is the limit.

b) Same question with $x_{n+1} = \frac{-1}{x_n + 1}$.

3. Let f be continuous on $[0, \infty)$ and let $|f(t) - f(0)| \leq Kt$ for $t > 0$ and some constant K .

Show that the sequence $c_n = n \int_0^\infty f(t) e^{-nt} dt$ converges and find its limit.

4. Let $f: [a, b] \rightarrow \mathbb{R}$ be continuous. Suppose that

$$\int_a^b f(x) x^n dx = 0 \quad \text{for } n = 0, 1, 2, \dots$$

Show that $f(x) = 0$ on $[a, b]$.

5. The gamma function $\Gamma(x)$ is defined for $x > 0$ by

$$\Gamma(x) = \int_0^{\infty} t^{x-1} e^{-t} dt \quad (*)$$

Show that $\log \Gamma(x)$ is a convex function.

(You may use the fact that the gamma function is infinitely differentiable and that its derivatives are obtained by differentiation under the integral sign in (*))

6. Find the radius of convergence of the series

(I) $\sum (x^{n^2})/n!$

(II) $\sum (x^{n^2})/(n!)^n$

7. Show that the series $\sum x\sqrt{n}/(1+n^2x^2)$ is convergent for every $x \in \mathbb{R}$. Discuss the uniform convergence of this series.

For $x \in \mathbb{R}$, let

$$f(x) = \sum x\sqrt{n}/(1+n^2x^2).$$

Is the function f continuous on \mathbb{R} ?

March 1993

ANALYSIS QUALIFYING EXAMINATION
(THE GRADING PROCEDURE)

PAUL NEVAI

Theorem: if x is the grade of a problem then $0 \leq x \leq 20$

Deadline: yesterday or today, whichever comes first

Please grade the enclosed solutions and then pass it on to the next person according to the schedule indicated on the envelope. Please mark your grades on (one of) the enclosed grading sheet which should be returned directly to me, that is, to Paul. Thanks!

Problem 1. Find the smallest constant $C > 0$ such that

$$(a + b + c + d + e)^2 \leq C (a^2 + b^2 + c^2 + d^2 + e^2)$$

for every $a \geq 0, b \geq 0, c \geq 0, d \geq 0$, and $e \geq 0$.

Problem 2. Find all sets on the real line where the series

$$\sum_{k=1}^{\infty} \frac{\cos kx}{k}$$

converges uniformly. You must not refer to any known result such as Abel's theorem. Instead, you should use direct estimates of the tail of the series. Among others, you may want to apply summation by parts.

Problem 3. Let $a_n = n \sum_{k=2n}^{\infty} \frac{(-1)^k}{k}$ for $n \in \mathbb{N}$. Prove that $\lim_{n \rightarrow \infty} a_n$ exists and find its value.

Problem 4. Prove that the sequence $\{a_n\}_{n=1}^{\infty}$ given by $a_n = \left(1 + \frac{1}{n}\right)^n$ is strictly increasing.

Problem 5. Let $f : \mathbb{R}^+ \mapsto \mathbb{R}$ be continuous and periodic with period 1. Prove that

$$\lim_{t \rightarrow \infty} \int_0^1 f(tx) dx \quad \text{exists and is equal to} \quad \int_0^1 f(y) dy.$$

Problem 6. For $n \in \mathbb{N}$, let $f_n : [0, 1] \mapsto \mathbb{R}$ be continuous functions, and assume that the series $\sum_{n=1}^{\infty} f_n$ converges uniformly in $[0, 1]$.

a) Prove that

$$\int_{\Delta} \left[\sum_{n=1}^{\infty} f_n(t) \right] dt = \sum_{n=1}^{\infty} \left[\int_{\Delta} f_n(t) dt \right]$$

for every interval $\Delta \subseteq [0, 1]$.

b) Give an example showing that

$$\left[\sum_{n=1}^{\infty} f_n(t) \right]' = \sum_{n=1}^{\infty} f_n'(t), \quad t \in [0, 1],$$

may fail even if $f_n'(t)$ exists for all $n \in \mathbb{N}$ and $t \in [0, 1]$.

Problem 7.

a) Give a complete statement of Dini's theorem about uniformly convergent monotonic sequences of functions.

b) Study the uniform convergence for $x \in \mathbb{R}$ of the sequence $\{s_n\}_{n=1}^{\infty}$ given by

$$s_n(x) = \frac{x^2 + 2x^{2n} - x^{2n+2}}{1 + x^{2n}}, \quad x \in \mathbb{R}.$$

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Four completely solved problems are sufficient for passing.

1. Show that the series $\sum_{n=1}^{\infty} (\sqrt{n+1} - 2\sqrt{n} + \sqrt{n-1})$ converges and find its sum.
2. Consider a sequence of real numbers $\{a_n\}_{n=1}^{\infty}$ such that $|a_n| \leq \frac{1}{n}$. Suppose that for every m the difference between the number of positive a_n and the number of negative a_n ($1 \leq n \leq m$) remains bounded as $m \rightarrow \infty$. Prove that $\sum_{n=1}^{\infty} a_n$ converges. (Hint: You may wish to use the summation by parts formula.)
3. Let f be defined on $E = \{x : x > 0\}$ by $f(x) = \sum_{k=1}^{\infty} a_k / (b_k + x)$ where $\sum_{k=1}^{\infty} |a_k| < \infty$ and $b_k > 0$ for $k = 1, 2, \dots$. Prove that f is indeed defined, and moreover continuous on E , and that $\lim_{x \rightarrow 0+} xf(x) = 0$. Is $f(x)$ continuously differentiable on E ?
4. Consider a polynomial $P_n(x) = 1 + x + \frac{x^2}{2!} + \dots + \frac{x^n}{n!}$, $n \geq 0$.
Prove that (a) $P_n(x)$ does not have real roots if n is even,
(b) $P_n(x)$ has exactly one real root if n is odd.
(Hint: For n even, look at $\inf\{P_n(x) : x \in \mathbb{R}\}$.)
5. Let $a > 0$ and $f(x)$ be continuously differentiable on $[0, a]$. Show that

$$|f(0)| \leq \frac{1}{a} \int_0^a |f(x)| dx + \int_0^a |f'(x)| dx.$$
6. Describe precisely the class of all functions which can be uniformly approximated on $[0, 1]$
 - (i) by polynomials of the form $P_n(x) = \sum_{k=0}^n a_k x^{2k}$,
 - (ii) by polynomials of the form $Q_n(x) = \sum_{k=0}^n a_k x^{2k+1}$.
7. Let a_0, a_1 and β be given, $0 < \beta < 1$. Let the sequence $\{a_n\}$ be defined by

$$a_{n+2} = \beta a_{n+1} + (1 - \beta) a_n, \quad \text{for } n \geq 0.$$

Show that $\{a_n\}$ converges, and find its limit.

Analysis Qualifying Exam, April 2, 1992.

Four complete solutions are sufficient to pass.

1. Let $Q_1(x) = 1$ and $Q_{n+1}(x) = (1/2)(Q_n(x)^2 + 1 - x^2)$ for every natural number n . Show that $Q_n(x)$ converges to $1 - |x|$ for all $x \in [-1, 1]$.

2. Show that if f is twice differentiable on the closed interval $[a-h, a+h]$, where $h > 0$, then there is a $\xi \in (a-h, a+h)$ such that

$$h^2 f''(\xi) = f(a+h) - 2f(a) + f(a-h).$$

3. Let $A = \mathbb{R} \setminus \{-n : n \in \mathbb{N}\}$, where \mathbb{R} denotes the set of all real numbers and \mathbb{N} is the set of all natural numbers.

(a) Show that the series

$$\sum_{n=1}^{\infty} \frac{(-1)^n}{x+n}$$

converges to a real-valued function f defined on A .

(b) Is the convergence uniform on A ? Prove your answer.

(c) Show that f is continuous on A .

4. A real-valued function f defined on the closed interval $[0, 1]$ is called two-to-one if for each real number y the set $\{x \in [0, 1] : f(x) = y\}$ is either empty or contains exactly two elements. Show that if f is two-to-one, then f is discontinuous at some $c \in [0, 1]$.

5. Show that every closed subset A of the real number line is a countable intersection of open sets.

6. Let g be a real-valued continuous function defined on $[0, \infty)$ such that $|g(t)| \leq e^t$ for all $t \in [0, \infty)$. Show that

(a)
$$G(x) = \int_0^{\infty} e^{-xt} g(t) dt$$

exists (as an improper Riemann integral) and finite for all $x > 1$, and

(b)
$$\lim_{x \rightarrow \infty} G(x) = 0.$$

Qualifying Exam in Analysis

Jan. 9, 1992

Four completely solved questions are sufficient for passing.

1. Show that every real sequence contains a monotone subsequence.

2. a) Prove that the sequence

$$s_n = \frac{1}{\sqrt{3n^2 + n}} + \frac{1}{\sqrt{3n^2 + n + 1}} + \frac{1}{\sqrt{3n^2 + n + 2}} + \cdots + \frac{1}{\sqrt{3n^2 + 3n}}$$

is convergent and find its limit.

b) Show that $f_n(x) = \frac{x^{2n} + 1}{x^{2n} + x^2 + 1}$ converges for every real x as $n \rightarrow \infty$, and find $\lim_n f_n(x)$.

3. Let the sequences (x_n) be defined by $x_1 = 1, x_{n+1} = \frac{2x_n + 3}{x_n + 1}$ for $n \geq 1$.

a) Prove that the sequence (x_n) converges and find its limits ℓ .

b) Show that there exist positive constants A, q such that $q < 1$ and $|x_n - \ell| \leq Aq^n$ for every n .

4. For any real c let $N(c)$ be the number of solutions of the equation $\sin x = cx$. Prove that

$$N(c) \sim \frac{2}{\pi} \frac{1}{|c|} \text{ as } c \rightarrow 0.$$

5. For what pairs of reals (α, β) the improper integral $\int_1^\infty \frac{\sin(x^\alpha)}{x^\beta} dx$ converges?

6. Let $f(x) = \sum_n \frac{1 - \cos x}{1 + x^2 + n^2 x^4}$ for x real.

a) Determine the intervals on which this series is uniformly convergent.

b) Is f continuous at 0?

Qualifying Examination in Analysis

October 3, 1991

Show your work. Calculators are not permitted. Four out of eight problems is sufficient for MS Pass; six out of eight for PhD Pass.

1. Show that

$$\lim_{n \rightarrow \infty} \left(\frac{1}{2} + \frac{n-1}{4n} + \frac{(n-1)(n-2)}{8n^2} + \dots + \frac{(n-1)(n-2) \dots 3 \cdot 2 \cdot 1}{2^n n^{n-1}} \right) = 1.$$

2. Let $f: \mathbb{R} \rightarrow \mathbb{R}$ be a function. Prove that the following two conditions are equivalent. (Either one may be taken as the definition of "measurable function".)

- (a) For all real numbers λ , the set $\{x \in \mathbb{R} : f(x) \geq \lambda\}$ is a measurable set.
- (b) There is a sequence f_n of simple functions (finite linear combinations of characteristic functions of measurable sets) that converges pointwise to f .

3. Use the definition of the Riemann integral (not properties of the Lebesgue integral) to prove the following version of the fundamental theorem of calculus: Let $F: [a, b] \rightarrow \mathbb{R}$ be differentiable, and suppose the derivative F' is bounded and Riemann integrable on $[a, b]$. Then

$$\int_a^b F'(x) dx = F(b) - F(a).$$

4. If a, b, c are positive and $a + b + c = 1$, show

$$\left(\frac{1}{a} - 1 \right) \left(\frac{1}{b} - 1 \right) \left(\frac{1}{c} - 1 \right) \geq 8.$$

When does equality hold?

5. Let $\varphi: \mathbb{R} \rightarrow \mathbb{R}$ be a function, and let $C = \{x \in \mathbb{R} : \varphi \text{ is continuous at } x\}$. Give examples where

- (a) $C = \mathbb{R}$,
- (b) $C = \emptyset$,
- (c) C and its complement $\mathbb{R} \setminus C$ are both dense.

6. Prove Schwartz's inequality: if $f, g \in L_2(\mathbb{R})$, then the product $fg \in L_1(\mathbb{R})$ and

$$\left| \int fg \right|^2 \leq \left(\int |f|^2 \right) \left(\int |g|^2 \right).$$

Su. 91
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Qualifying Examination in Analysis
Summer, 1991

This is a combined M.S. and Ph.D. Qualifying Examination. Four problems done correctly is sufficient for a M.S. pass and six problems done correctly is sufficient for a Ph.D. pass. Please show all of your work.

1. Let (s_n) be a sequence of real numbers such that there is a real number s with $s_n \leq s$ for all n . Also suppose that for all $n \geq 1$, $s_{n+1} \geq \frac{1}{2}(s_n + s)$. Prove that the sequence (s_n) converges.

2. Let (a_n) be the sequence given by $a_1 = 1$ and $a_{n+1} = \frac{1}{2}(a_n + \frac{3}{a_n})$. Show that this sequence converges and find its limiting value.

3. Consider the series $\sum_{n=1}^{\infty} \frac{1}{1+n^2} z^n$. Show that this series converges for all complex numbers z with $|z| \leq 1$, and that it does not converge for any complex number z with $|z| > 1$.

4. Let F be a closed subset of the real numbers and let K be a compact subset of the real numbers. Let A be the set of all sums $f + k$ where f is in F and k is in K . Show that A is a closed subset of the real numbers.

5. Let $f: [a, b] \rightarrow \mathbb{R}$ be a continuous function whose derivative Df exists on (a, b) . Show that f has bounded variation if Df is bounded, but that f can be of bounded variation without Df being bounded.

7. Prove Dini's theorem: Let (f_n) be a sequence of continuous functions on $[0, 1]$ such that, for each $x \in [0, 1]$,

$$f_n(x) \geq f_{n+1}(x), \quad n = 1, 2, \dots \text{ and}$$

$$\lim_{n \rightarrow \infty} f_n(x) = 0.$$

Then f_n converges to 0 uniformly on $[0, 1]$.

8. Prove Cauchy's mean value theorem: Suppose functions f and g are continuous on the closed interval $[a, b]$ and differentiable on the open interval (a, b) . Then there is $u \in (a, b)$ such that

$$[f(b) - f(a)] g'(u) = [g(b) - g(a)] f'(u).$$

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6. Let f be a continuous function on $[0, 1]$ and suppose $f(x) \geq 0$ for all x in $[0, 1]$. Show that $\int_0^1 f^n(x) dx$ converges as $n \rightarrow \infty$ and describe exactly when this limit is ∞ .
7. Let f and g be Lebesgue integrable functions on $[0, 1]$ and suppose that for all integer values of n , $\int_0^1 f(x) \exp(nx) dx = \int_0^1 g(x) \exp(nx) dx$. Show that f must be equal to g on $[0, 1]$ almost everywhere with respect to Lebesgue measure.
8. Suppose f is a Lebesgue integrable function on the real line \mathbb{R} and that $\int_{\mathbb{R}} |f(x)|^2 dx$ is finite. Show that for all p , $1 \leq p \leq 2$, $\int_{\mathbb{R}} |f(x)|^p dx$ is finite too.

WJ 91

WJ 91

Analysis Ph.D. Qualifying Examination

January 11, 1991

Show your work. Solving correctly four out of the seven problems is sufficient to pass the examination.

1. Suppose (a_n) is the sequence of real numbers given by $a_1 > 0$ and $a_{n+1} = (1 + a_n)^{1/2}$. Show that for all values of a_1 , the sequence (a_n) converges to a real number.

2. For a vector x in \mathbb{R}^n , let (x_1, \dots, x_n) be its usual coordinates. Consider the metric $d(x, y)$ on \mathbb{R}^n given by

$$d(x, y) = d((x_1, \dots, x_n), (y_1, \dots, y_n)) = \left(\sum_{i=1}^n (x_i - y_i)^2 \right)^{1/2}.$$

Show that a sequence of vectors $(x(m) : m = 1, 2, 3, \dots)$ converges in the topology given by the metric d if and only if for each $i = 1, \dots, n$, the sequence of coordinates $(x_i(m) : m = 1, 2, 3, \dots)$ converges.

3. Let a and b be positive numbers, $a > b$, and let (x_n) be a sequence of real numbers. Assume that $\lim_{n \rightarrow \infty} n^a x_n = 1$. Show that

$\sum_{n=1}^{\infty} (x_n - x_{n+1}) n^b$ converges.

4. Let A and B be closed subsets of the closed interval $[0,1]$. Show that A and B are disjoint if and only if $\inf\{|x - y| : x \in A, y \in B\} > 0$.
5. Consider a function $F: \mathbb{R} \rightarrow \mathbb{R}$ which has a continuous derivative DF such that $\lim_{|x| \rightarrow \infty} DF(x) = 0$. Show that F is uniformly continuous on \mathbb{R} .
6. Show that for every continuous function $f: [0,1] \rightarrow \mathbb{R}$, and $\epsilon > 0$, there exist pairs of real numbers $(a_1, b_1), \dots, (a_n, b_n)$ such that for all $x \in [0,1]$,

$$|f(x) - \sum_{i=1}^n a_i \exp(b_i x)| < \epsilon.$$

7. Let m denote the Lebesgue measure on \mathbb{R} . Let (f_n) be a sequence of real-valued measurable functions on \mathbb{R} such that $\sum_{n=1}^{\infty} m(\{x \in \mathbb{R} : |f_n(x)| > n\})$ converges to a real number. Show that $\lim_{n \rightarrow \infty} \frac{f_n(x)}{n^2} = 0$ almost everywhere with respect to m .

Qualifying Master's Examination in Analysis

Thursday, April 4, 1991

SP 91
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Five correctly done problems on this examination are sufficient to pass. Please show all of your work and explain your arguments in detail.

1. Consider the function $f(x) = x^2 \exp(-x^2)$. Show that f is uniformly continuous on \mathbb{R} .

2. Let $f(x)$ be a function on \mathbb{R} which is differentiable at 0. Assume that $f(x) = f(-x)$ for all real numbers x . Show that the derivative of f at 0 is equal to 0.

3. Prove that for all real numbers $x \neq 0$, there is a real number y such that

$$\frac{\sin(x) - x + \frac{x^3}{6}}{x^5} = \frac{\cos(y)}{120}.$$

4. Consider the sequence (a_n) given by $a_1 = 3$ and $a_{n+1} = \frac{1}{2 + a_n}$. Show that the sequence converges and calculate its limit.

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5. Given increasing sequences (a_n) and (b_n) with $\lim_{n \rightarrow \infty} \frac{a_n}{b_n} = 0$, show that there exists a sequence (c_n) such that $\sum_{n=1}^{\infty} c_n a_n$ converges but $\sum_{n=1}^{\infty} c_n b_n$ diverges.
6. Consider the series $S(x) = \sum_{n=1}^{\infty} a_n \cos(2\pi n x)$. Assume that $\sum_{n=1}^{\infty} n|a_n| < \infty$. Show that then S is a well-defined continuous function on \mathbb{R} with a derivative that is also a continuous function on \mathbb{R} .
7. Consider the series $S(x) = \sum_{n=1}^{\infty} n x^n$. Show that $\lim_{x \rightarrow 1^-} (1-x)^2 S(x)$ exists and calculate its value.
8. Show that if $f_n(x) = 4^n x^n (1-x)^n$, then $\lim_{n \rightarrow \infty} \int_0^1 f_n(x) dx = 0$.

ANALYSIS

August 21, 1990

Show your work. Calculators are not permitted. Solving four out of six problems is sufficient to pass the exam.

1. Let $0 < a < b < \infty$. Define $x_1 = a$, $x_2 = b$, $x_{n+2} = (x_n + x_{n+1})/2$. Does (x_n) converge? If so, to what limit?
2. Consider two metrics on \mathbb{R}^n given by:

$$\rho((x_1, \dots, x_n), (y_1, \dots, y_n)) = \left(|x_1 - y_1|^2 + \dots + |x_n - y_n|^2 \right)^{1/2}$$

$$\sigma((x_1, \dots, x_n), (y_1, \dots, y_n)) = \max \{ |x_1 - y_1|, \dots, |x_n - y_n| \}.$$

Show that these two metrics generate the same topology.

3. Evaluate these limits:

$$(a) \lim_{x \downarrow 0} x^x$$

$$(b) \lim_{x \downarrow 0} x^{\log x}$$

$$(c) \lim_{x \downarrow 1} (\log x)^x$$

$$(d) \lim_{x \downarrow 1} (\log x)^{\log x}.$$

4. Let $E \subseteq \mathbb{R}$ be a set with $\lambda(E) = 0$. (Lebesgue measure on \mathbb{R} is denoted by λ .) Show that there exists a countable family \mathcal{J} of open intervals such that each point of E belongs to infinitely many members of \mathcal{J} and

$$\sum_{I \in \mathcal{J}} \lambda(I) < \infty.$$

5. Suppose f is Riemann integrable over the interval $[a, b]$. Show that $|f|$ is also Riemann integrable over $[a, b]$, and

$$\left| \int_a^b f(x) dx \right| \leq \int_a^b |f(x)| dx.$$

6. Suppose $f: \mathbb{R} \rightarrow \mathbb{R}$ is uniformly continuous and $f \in L_p$, for some p with $0 < p < \infty$. Show that $\lim_{x \rightarrow \infty} f(x) = \lim_{x \rightarrow -\infty} f(x) = 0$.