2015 Rasor-Bareis examination problems

1. Prove that $\cos(\pi/2^n)$ is irrational for all integer $n \geq 2$.

2. Let m and n be positive integers such that $m/n < \sqrt{2}$. Prove that, in fact, $m/n < \sqrt{2}\left(1 - \frac{1}{4n^2}\right)$.

3. An equiangular 2015-gon P is inscribed in a circle. Prove that P is regular. (A polygon is said to be *equiangular* if all its angles are equal. It is *regular* if, additionally, the lengths of all its sides are equal.)

4. Let $n \in \mathbb{N}$ and let $a_1, a_2, \ldots, a_n; b_1, b_2, \ldots, b_n$ be all the integers from 1 to $2n$ ordered so that $a_1 < \cdots < a_n$ and $b_1 > \cdots > b_n$. Prove that $\sum_{i=1}^{n} |a_i - b_i| = n^2$.

5. The points on the sides of an equilateral triangle are colored with two colors. Prove that there are three points P, Q, R of the same color such that $\triangle PQR$ is a right triangle.

6. Evaluate $\displaystyle\int_{-1}^{1} \frac{dx}{1 + x^3 + \sqrt{1 + x^6}}$.