2016 Rasor-Bareis examination problems

1. Given a set A of real numbers, we are allowed to replace any two distinct numbers a, b from A by $\frac{a+b}{\sqrt{2}}$ and $\frac{a-b}{\sqrt{2}}$. From initial set $S = \{1, 2, 4\}$, can we apply that operation several times to obtain the set $S' = \left\{\sqrt{2}, 2\sqrt{2}, 3\right\}$?

2. There are 2016 points in the plane such that any triangle with the vertices at three of those points has area at most 1. Prove that all these points are contained in a triangle of area 4.

3. Let $n \geq 2$ and let $a_1, \ldots, a_n > 0$; prove that $\dfrac{a_1^2}{a_2} + \dfrac{a_2^2}{a_3} + \cdots + \dfrac{a_{n-1}^2}{a_n} \geq 4(a_1 - a_n)$.

4. Let (a_n) be a bounded increasing sequence of positive real numbers. Prove that

$$\sum_{n=1}^{\infty} \left(1 - \frac{a_n}{a_{n+1}}\right) < \infty.$$

5. Prove that there are infinitely many positive integers not representable in the form $n^2 + p$, where n is an integer and p is prime.

6. Let α, β, γ be the angles of a triangle. If $\sin\alpha$, $\sin\beta$, $\sin\gamma$ are all rational, prove that $\cos\alpha$, $\cos\beta$, $\cos\gamma$ are also rational.