1. Prove that \( a_n = \cos(\pi/2^n) \) is irrational for all integer \( n \geq 2 \).

Solution. We use induction on \( n \). For \( n = 2 \), \( a_2 = \cos(\pi/4) = 1/\sqrt{2} \) is irrational. The formula \( a_n = 2a_{n+1}^2 - 1 = \) shows that if \( a_{n+1} \) is rational, then \( a_n \) is also rational; hence if, by induction, \( a_n \) is irrational, then \( a_{n+1} \) is also irrational.

2. Let \( m \) and \( n \) be positive integers such that \( m/n < \sqrt{2} \). Prove that \( m/n < \sqrt{2} \left(1 - \frac{1}{4n^2}\right) \).

Solution. We write a sequence of equivalent inequalities:

\[
\frac{m}{n} < \sqrt{2} \left(1 - \frac{1}{4n^2}\right) \iff 1 - \frac{m}{\sqrt{2}n} > \frac{1}{4n^2} \iff \frac{\sqrt{2}n - m}{\sqrt{2}n} > \frac{1}{4n^2} \iff \frac{2n^2 - m^2}{(\sqrt{2}n + m)\sqrt{2}n} > \frac{1}{4n^2}.
\]

But since \( m < \sqrt{2}n \), so \( m^2 < 2n^2 \), and therefore \( 2n^2 - m^2 \geq 1 \), we see that

\[
\frac{2n^2 - m^2}{(\sqrt{2}n + m)\sqrt{2}n} > \frac{1}{4n^2}.
\]

Another solution. Assume that \( m/n \geq \sqrt{2} \left(1 - \frac{1}{4n^2}\right) \). Then

\[
\sqrt{2} \left(1 - \frac{1}{4n^2}\right) \leq \frac{m}{n} < \sqrt{2}, \quad \text{so} \quad 2 \left(1 - \frac{1}{4n^2}\right)^2 \leq \frac{m^2}{n^2} < 2, \quad \text{so} \quad 2 - \frac{1}{n^2} + \frac{1}{8n^4} \leq \frac{m^2}{n^2} < 2,
\]

which is impossible since \( n^2 \) and \( m^2 \) are both integers.

3. An equiangular 2015-gon \( P \) is inscribed in a circle. Prove that \( P \) is regular.

Solution. We only need to prove that the sides of \( P \) have the same length. Let the vertices of \( P \) be \( A_1, A_2, \ldots, A_{2015} \), and for each \( i \), let \( |A_iA_{i+1}| = a_i \). (We take \( i \) modulo 2015, that is, assume that \( A_{2016} = A_1, A_{2017} = A_2 \), etc., so that \( a_{2016} = a_1 \).) Then, for any \( i \), we are given that \( \angle A_iA_{i+1}A_{i+2} = \angle A_{i+3}A_{i+2}A_{i+1} \), and \( \angle A_{i+1}A_iA_{i+2} = \angle A_{i+2}A_iA_{i+3} \). Since these are angles subtended by the same chord, the triangles \( \triangle A_iA_{i+1}A_{i+2} \) and \( \triangle A_{i+3}A_iA_{i+1} \) are congruent, so \( a_i = a_{i+2} \). Hence, \( a_1 = a_3 = \cdots = a_{2015} = a_2 = a_4 = \cdots = a_{2014} \).

Another solution. Let the vertices of \( P \) be \( A_1, A_2, \ldots, A_{2015} \), and let the angles of the polygon be all equal to \( \beta \). For each \( i \), the triangle \( OA_iA_{i+1} \) is isosceles, thus \( \angle O A_i A_{i+1} = \angle O A_i A_{i+1} \); let us denote this angle by \( \alpha_i \). Then, for any \( i \), \( \alpha_i + \alpha_{i+1} = \alpha_{i+1} + \alpha_{i+2} = \beta \), so \( \alpha_i = \beta - \alpha_{i+1} = \alpha_{i+2} \), and hence, \( \alpha_1 = \alpha_3 = \cdots = \alpha_{2015} = \alpha_2 = \cdots = \alpha_{2014} \). It follows that all the triangles \( \triangle O A_i A_{i+1} \) are congruent, and so all the sides \( A_iA_{i+1} \) of the polygon have equal length.

4. Let \( n \in \mathbb{N} \) and let \( a_1, a_2, \ldots, a_n; b_1, b_2, \ldots, b_n \) be all the integers between 1 and \( 2n \) ordered so that \( a_1 < \cdots < a_n \) and \( b_1 > \cdots > b_n \). Prove that \( \sum_{i=1}^{n} |a_i - b_i| = n^2 \).

Solution. Let us place the numbers \( a_i, b_i \) in their natural order, say, \( b_n, a_1, a_2, b_{n-1}, \ldots, a_n, b_1 \), meaning that \( b_n = 1, a_1 = 2, a_2 = 3, b_{n-1} = 4, \ldots, b_1 = 2n \). The “standard” ordering, where \( a_i = i \) and \( b_i = 2n - i + 1, i = 1, \ldots, n \), then takes the form \( a_1, a_2, \ldots, a_n, b_n, \ldots, b_3, b_2, b_1 \). For this ordering we have

\[
\sum_{i=1}^{n} |a_i - b_i| = (2n - 1) + (2n - 3) + \cdots + 1 = n^2.
\]
Any other ordering can be obtained from the standard one by “pushing” some of $a_k$ to the right, that is, by applying a sequence of transpositions of the form

$$\tau_{k,l} : \ldots, a_k, b_l, \ldots \mapsto \ldots, b_l, a_k, \ldots$$

(That is, if $a_k = m$ and $b_l = m + 1$ for some $m$, then after the application of $\tau_{k,l}$ we have $b_l = m$, $a_k = m + 1$, and all other $a_i$ and $b_j$ keep their values.) If we show that any such transposition preserves the sum $\sum_{i=1}^n |a_i - b_i|$, we are done. And indeed, under the transposition $\tau_{k,l}$, for any $i \neq k, l$, the difference $a_i - b_i$ remains intact. If $k < l$, then $b_l, a_k < a_i, b_k$, and the transposition $\tau_{k,l}$ decrements $|a_k - b_k|$ by 1 and increments $|a_l - b_l|$ by 1. If $l < k$, then $a_l, b_k < a_k, b_l$, and the transposition $\tau_{k,l}$ increments $|a_k - b_k|$ by 1 and decrements $|a_l - b_l|$ by 1. Finally, if $l = k$, the difference $|a_k - b_k|$ remains equal to 1.

Another solution. For any $i$ one has $|a_i - b_i| = \max(a_i, b_i) - \min(a_i, b_i)$. Now, $a_1 < a_1, \ldots, a_{i-1}$ and $b_i > b_{i+1}, \ldots, b_n$, so $\max(a_i, b_i)$ is greater than the $n$ integers $a_1, \ldots, a_{i-1}, \min(a_i, b_i), b_{i+1}, \ldots, b_n$. Hence, $\max(a_i, b_i)$ is in the interval $\{n + 1, \ldots, 2n\}$. Similarly, $\min(a_i, b_i)$ is less than the $n$ integers $a_{i+1}, \ldots, a_n$, $\max(a_i, b_i), b_1, \ldots, b_{i-1}$ and so, lies in the interval $\{1, \ldots, n\}$. Thus, the set $\{\min(a_i, b_i), i = 1, \ldots, n\}$ is $\{1, \ldots, n\}$ and the set $\{\max(a_i, b_i), i = 1, \ldots, n\}$ is $\{n + 1, \ldots, 2n\}$, and so,

$$\sum_{i=1}^n |a_i - b_i| = \sum_{i=1}^n (\max(a_i, b_i) - \min(a_i, b_i)) = \sum_{i=1}^n \max(a_i, b_i) - \sum_{i=1}^n \min(a_i, b_i)$$

$$= \sum_{k=n+1}^{2n} k - \sum_{m=1}^n m = \sum_{m=1}^n (n + m) - \sum_{m=1}^n m = \sum_{m=1}^n (n + m - m) = \sum_{m=1}^n n = n^2.$$

5. The points on the sides of an equilateral triangle are colored with two colors. Prove that there are three points $P, Q, R$ of the same color such that $\triangle PQR$ is a right triangle.

Solution. Let the points be colored red and blue, and assume, by the way of contradiction, that the statement is wrong. Let $A, B, C$ be the vertices of the triangle, and let $C_1, C_2, A_1, A_2, B_1, B_2$ be the points on the sides of the triangle subdividing each of the sides to three equal parts. Then the lines $(B_1C_1)$ and $(A_2C_2)$ are orthogonal to the line $(AB), (C_1A_1)$ and $(B_2A_2)$ to the line $(BC), (C_1B_1)$ and $(A_2B_2)$ to the line $(CA)$, which provides us with a bunch of right triangles with vertices on the sides of $\triangle ABC$.

We claim that any two opposite vertices of the hexagon $C_1C_2A_1A_2B_1B_2$ are colored differently. Indeed, assume that the vertices $C_1$ and $A_2$ are both red. If one of the vertices $C_2, A_1, B_1, B_2$, say $C_2$, is red, then $\triangle C_1C_2A_2$ is right-red (that is, right with red vertices). But if the vertices $C_2, A_1, B_1, B_2$ are all blue, then $\triangle A_1B_1B_2$ is right-blue.

It now follows that in at least one of the pairs $(C_1, C_2)$, $(A_1, A_2)$, or $(B_1, B_2)$ the points are colored differently. Without loss of generality, assume that $C_1$ is red and $C_2$ is blue. Then $A_2$ is blue and $B_1$ is red. Now, if $A$ is red, then $\triangle AC_1B_1$ is right-red; if $A$ is blue, then $\triangle AC_2B_2$ is right-blue.

Another solution. The following proof works in the case the colored triangle is not necessarily equilateral, but an arbitrary acute or right triangle. (For an obtuse triangle, a simple counterexample exists: the longest side is of one color, and the other two sides are of the other color.)

Assume that the statement is wrong. Consider two cases:

Case 1: One of the sides (say, $AB$) is monochromatic (say, red) with at most one point $X$ of the other (blue) color. Then all points on the sides $AC$ and $BC$, except $A$, $B$, and, maybe, one more point corresponding to $X$, are blue: if there is a red point $P \neq A, B$ on $AC$ or $BC$ whose orthogonal projection $Q$ on $AB$ is distinct from $X$ and so, is red, then for any other red point $R$ on $AB$ the triangle $\triangle PQR$ is right-red. But then there are many right-blue triangles with vertices on the sides $AC$ and $BC$. 

\[ \text{A counterexample} \]
Case 2: Each side of the triangle contains at least two points of the same color. Then for any non-vertex point $P$ of the triangle (say, on the side $AB$) its orthogonal projection $Q$ on, say, the side $AC$ must be of different color: if both $P$ and $Q$ were, say, red, then for any other red point $R$ on $AC$ the triangle $\triangle PQR$ would be right-red.

But, if $P$ is red and $Q$ is blue, then, for the same reason, the orthogonal projection $R$ of $Q$ on the side $BC$ is red, and the orthogonal projection $P'$ of $R$ on the side $AB$ is blue. Now, to get a contradiction, it is enough to show that there exists a point $P \in AB$ such that $P' = P$. And indeed, if $P$ is chosen close to vertex $A$, then $P'$ is to the right of $P$; if $P$ is taken close to $B$, then $P'$ is to the left of $P$; so, since $P'$ depends on $P$ continuously, there must be a point $P \in AB$ such that $P' = P$.

6. Evaluate $\int_{-1}^{1} \frac{dx}{1 + x^3 + \sqrt{1 + x^6}}$.

Solution.

$$
\int_{-1}^{1} \frac{dx}{1 + x^3 + \sqrt{1 + x^6}} = \int_{-1}^{0} \frac{dx}{1 + x^3 + \sqrt{1 + x^6}} + \int_{0}^{1} \frac{dx}{1 + x^3 + \sqrt{1 + x^6}} \\
= \int_{0}^{1} \frac{dx}{1 - x^3 + \sqrt{1 + x^6}} + \int_{0}^{1} \frac{dx}{1 + x^3 + \sqrt{1 + x^6}} \\
= \int_{0}^{1} \left( \frac{1}{1 - x^3 + \sqrt{1 + x^6}} + \frac{1}{1 + x^3 + \sqrt{1 + x^6}} \right) dx \\
= \int_{0}^{1} \frac{2 + 2\sqrt{1 + x^6}}{1 + 2\sqrt{1 + x^6} + (1 + x^6) - x^6} dx = \int_{0}^{1} \frac{2 + 2\sqrt{1 + x^6}}{2 + 2\sqrt{1 + x^6}} dx = \int_{0}^{1} dx = 1.
$$