2015 Rasor-Bareis exam solutions

1. *Prove that $a_n = \cos(\pi/2^n)$ is irrational for all integer $n \geq 2$.*

Solution. We use induction on n. For $n = 2$, $a_2 = \cos(\pi/4) = 1/\sqrt{2}$ is irrational. The formula $a_n = 2a_{n+1}^2 - 1 =$ shows that if a_{n+1} is rational, then a_n is also rational; hence if, by induction, a_n is irrational, then a_{n+1} is also irrational.

2. *Let m and n be positive integers such that $m/n < \sqrt{2}$. Prove that $m/n < \sqrt{2}\big(1 - \frac{1}{4n^2}\big)$.*

Solution. We write a sequence of equivalent inequalities:

$$\frac{m}{n} < \sqrt{2}\Big(1 - \frac{1}{4n^2}\Big) \quad \text{iff} \quad 1 - \frac{m}{\sqrt{2}n} > \frac{1}{4n^2} \quad \text{iff} \quad \frac{\sqrt{2}n - m}{\sqrt{2}n} > \frac{1}{4n^2} \quad \text{iff} \quad \frac{2n^2 - m^2}{(\sqrt{2}n + m)\sqrt{2}n} > \frac{1}{4n^2}.$$

But since $m < \sqrt{2}n$, so $m^2 < 2n^2$, and therefore $2n^2 - m^2 \geq 1$, we see that

$$\frac{2n^2 - m^2}{(\sqrt{2}n + m)\sqrt{2}n} > \frac{1}{(\sqrt{2}n + \sqrt{2}n)\sqrt{2}n} = \frac{1}{4n^2}.$$

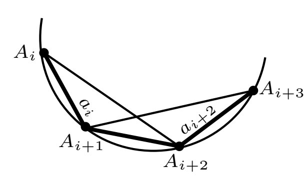
Another solution. Assume that $m/n \geq \sqrt{2}\big(1 - \frac{1}{4n^2}\big)$. Then

$$\sqrt{2}\Big(1 - \frac{1}{4n^2}\Big) \leq \frac{m}{n} < \sqrt{2}, \quad \text{so} \quad 2\Big(1 - \frac{1}{4n^2}\Big)^2 \leq \frac{m^2}{n^2} < 2, \quad \text{so} \quad 2 - \frac{1}{n^2} + \frac{1}{8n^4} \leq \frac{m^2}{n^2} < 2,$$
$$\text{so} \quad 2n^2 - 1 + \frac{1}{8n^2} \leq m^2 < 2n^2, \quad \text{so} \quad 2n^2 - 1 < m^2 < 2n^2,$$

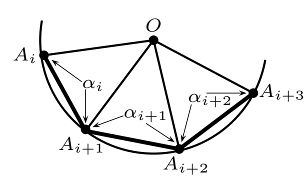
which is impossible since n^2 and m^2 are both integers.

3. *An equiangular 2015-gon P is inscribed in a circle. Prove that P is regular.*

Solution. We only need to prove that the sides of P have the same length. Let the vertices of P be $A_1, A_2, \ldots, A_{2015}$, and for each i, let $|A_iA_{i+1}| = a_i$. (We take i modulo 2015, that is, assume that $A_{2016} = A_1$, $A_{2017} = A_2$, etc., so that $a_{2016} = a_1$.) Then, for any i, we are given that $\angle A_iA_{i+1}A_{i+2} = \angle A_{i+3}A_{i+2}A_{i+1}$, and $\angle A_{i+1}A_iA_{i+2} = \angle A_{i+2}A_{i+3}A_{i+1}$ since these are angles subtended by the same chord, so the triangles $\triangle A_iA_{i+1}A_{i+2}$ and $\triangle A_{i+3}A_{i+2}A_{i+1}$ are congruent, so $a_i = a_{i+2}$. Hence, $a_1 = a_3 = \cdots = a_{2015} = a_2 = a_4 = \cdots = a_{2014}$.



Another solution. Let the vertices of P be $A_1, A_2, \ldots, A_{2015}$, and let the angles of the polygon be all equal to β. For each i, the triangle OA_iA_{i+1} is isosceles, thus $\angle OA_{i+1}A_i = \angle OA_{i+1}A_i$; let us denote this angle by α_i. Then, for any i, $\alpha_i + \alpha_{i+1} = \alpha_{i+1} + \alpha_{i+2} = \beta$, so $\alpha_i = \beta - \alpha_{i+1} = \alpha_{i+2}$, and, hence, $\alpha_1 = \alpha_3 = \ldots = \alpha_{2015} = \alpha_2 = \ldots = \alpha_{2014}$. It follows that all the triangles $\triangle OA_iA_{i+1}$ are congruent, and so all the sides A_iA_{i+1} of the polygon have equal length.



4. *Let $n \in \mathbb{N}$ and let $a_1, a_2, \ldots, a_n; b_1, b_2, \ldots, b_n$ be all the integers between 1 and $2n$ ordered so that $a_1 < \cdots < a_n$ and $b_1 > \cdots > b_n$. Prove that $\sum_{i=1}^n |a_i - b_i| = n^2$.*

Solution. Let us place the numbers a_i, b_j in their natural order, say, $b_n, a_1, a_2, b_{n-1}, \ldots, a_n, b_1$, meaning that $b_n = 1$, $a_1 = 2$, $a_2 = 3$, $b_{n-1} = 4$, $\ldots$, $b_1 = 2n$. The "standard" ordering, where $a_i = i$ and $b_i = 2n - i + 1$, $i = 1, \ldots, n$, then takes the form $a_1, a_2, a_3, \ldots, a_n, b_n, \ldots, b_3, b_2, b_1$. For this ordering we have

$$\sum_{i=1}^{n} |a_i - b_i| = (2n-1) + (2n-3) + \cdots + 1 = n^2.$$

Any other ordering can be obtained from the standard one by "pushing" some of a_k to the right, that is, by applying a sequence of transpositions of the form

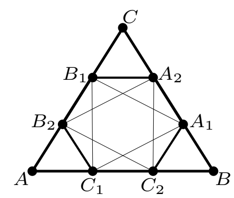
$$\tau_{k,l} : \ldots, a_k, b_l, \ldots \mapsto \ldots, b_l, a_k, \ldots$$

(That is, if $a_k = m$ and $b_l = m+1$ for some m, then after the application of $\tau_{k,l}$ we have $b_l = m$, $a_k = m+1$, and all other a_i and b_j keep their values.) If we show that any such transposition preserves the sum $\sum_{i=1}^{n} |a_i - b_i|$, we are done. And indeed, under the transposition $\tau_{k,l}$, for any $i \neq k, l$, the difference $a_i - b_i$ remains intact. If $k < l$, then $b_l, a_k < a_l, b_k$, and the transposition $\tau_{k,l}$ decrements $|a_k - b_k|$ by 1 and increments $|a_l - b_l|$ by 1. If $l < k$, then $a_l, b_k < a_k, b_l$, and the transposition $\tau_{k,l}$ increments $|a_k - b_k|$ by 1 and decrements $|a_l - b_l|$ by 1. Finally, if $l = k$, the difference $|a_k - b_k|$ remains equal to 1.

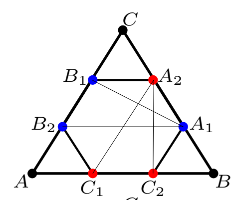
Another solution. For any i one has $|a_i - b_i| = \max(a_i, b_i) - \min(a_i, b_i)$. Now, $a_i > a_1, \ldots, a_{i-1}$ and $b_i > b_{i+1}, \ldots, b_n$, so $\max(a_i, b_i)$ is greater than the n integers $a_1, \ldots, a_{i-1}, \min(a_i, b_i), b_{i+1}, \ldots, b_n$. Hence, $\max(a_i, b_i)$ is in the interval $\{n+1, \ldots, 2n\}$. Similarly, $\min(a_i, b_i)$ is less than the n integers $a_{i+1}, \ldots, a_n$, $\max(a_i, b_i), b_1, \ldots, b_{i-1}$ and so, lies in the interval $\{1, \ldots, n\}$. Thus, the set $\{\min(a_i, b_i),\ i = 1, \ldots, n\}$ is $\{1, \ldots, n\}$ and the set $\{\max(a_i, b_i),\ i = 1, \ldots, n\}$ is $\{n+1, \ldots, 2n\}$, and so,

$$\begin{aligned}\sum_{i=1}^{n} |a_i - b_i| &= \sum_{i=1}^{n} \big(\max(a_i, b_i) - \min(a_i, b_i)\big) = \sum_{i=1}^{n} \max(a_i, b_i) - \sum_{i=1}^{n} \min(a_i, b_i) \\ &= \sum_{k=n+1}^{2n} k - \sum_{m=1}^{n} m = \sum_{m=1}^{n} (n+m) - \sum_{m=1}^{n} m = \sum_{m=1}^{n} (n+m-m) = \sum_{m=1}^{n} n = n^2.\end{aligned}$$

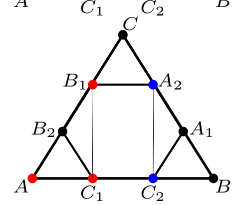
5. *The points on the sides of an equilateral triangle are colored with two colors. Prove that there are three points P, Q, R of the same color such that $\triangle PQR$ is a right triangle.*



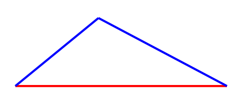
Solution. Let the points be colored red and blue, and assume, by the way of contradiction, that the statement is wrong. Let A, B, C be the vertices of the triangle, and let $C_1, C_2, A_1, A_2, B_1, B_2$ be the points on the sides of the triangle subdividing each of the sides to three equal parts. Then the lines (B_1C_1) and (A_2C_2) are orthogonal to the line (AB), (C_1A_1) and (B_2A_2) to the line (BC), (C_1B_1) and (A_2B_2) to the line (CA), which provides us with a bunch of right triangles with vertices on the sides of $\triangle ABC$.



We claim that any two opposite vertices of the hexagon $C_1C_2A_1A_2B_1B_2$ are colored differently. Indeed, assume that the vertices C_1 and A_2 are both red. If one of the vertices C_2, A_1, B_1, B_2, say C_2, is red, then $\triangle C_1C_2A_2$ is right-red (that is, right with red vertices). But if the vertices C_2, A_1, B_1, B_2 are all blue, then $\triangle A_1B_1B_2$ is right-blue.



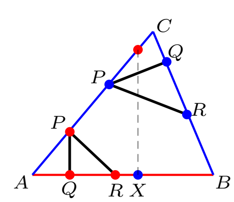
It now follows that in at least one of the pairs (C_1, C_2), (A_1, A_2), or (B_1, B_2) the points are colored differently. Without loss of generality, assume that C_1 is red and C_2 is blue. Then A_2 is blue and B_1 is red. Now, if A is red, then $\triangle AC_1B_1$ is right-red; if A is blue, then $\triangle AC_2A_2$ is right-blue.



A counterexample

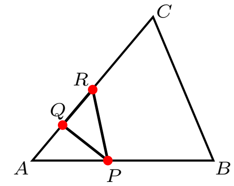
Another solution. The following proof works in the case the colored triangle is not necessarily equilateral, but an arbitrary acute or right triangle. (For an obtuse triangle, a simple counterexample exists: the longest side is of one color, and the other two sides are of the other color.)

Assume that the statement is wrong. Consider two cases:

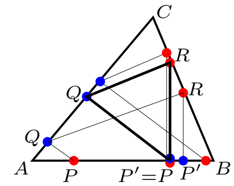


Case 1: One of the sides (say, AB) is monochromatic (say, red) with at most one point X of the other (blue) color. Then all points on the sides AC and BC, except A, B, and, maybe, one more point corresponding to X, are blue: if there is a red point $P \neq A, B$ on AC or BC whose orthogonal projection Q on AB is distinct from X and so, is red, then for any other red point R on AB the triangle $\triangle PQR$ is right-red. But then there are many right-blue triangles with vertices on the sides AC and BC.

Case 2: Each side of the triangle contains at least two points of the same color. Then for any non-vertex point P of the triangle (say, on the side AB) its orthogonal projection Q on, say, the side AC must be of different color: if both P and and Q were, say, red, then for any other red point R on AC the triangle $\triangle PQR$ would be right-red.



But, if P is red and Q is blue, then, for the same reason, the orthogonal projection R of Q on the side BC is red, and the orthogonal projection P' of R on the side AB is blue. Now, to get a contradiction, it is enough to show that there exists a point $P \in AB$ such that $P' = P$. And indeed, if P is chosen close to vertex A, then P' is to the right of P; if P is taken close to B, then P' is to the left of P; so, since P' depends on P continuously, there must be a point $P \in AB$ such that $P' = P$.



6. *Evaluate* $\displaystyle\int_{-1}^{1} \frac{dx}{1+x^3+\sqrt{1+x^6}}$.

Solution.

$$\begin{aligned}
\int_{-1}^{1} \frac{dx}{1+x^3+\sqrt{1+x^6}} &= \int_{-1}^{0} \frac{dx}{1+x^3+\sqrt{1+x^6}} + \int_{0}^{1} \frac{dx}{1+x^3+\sqrt{1+x^6}} \\
&= \int_{0}^{1} \frac{dx}{1-x^3+\sqrt{1+x^6}} + \int_{0}^{1} \frac{dx}{1+x^3+\sqrt{1+x^6}} \\
&= \int_{0}^{1} \Big(\frac{1}{1-x^3+\sqrt{1+x^6}} + \frac{1}{1+x^3+\sqrt{1+x^6}}\Big) dx \\
&= \int_{0}^{1} \frac{2+2\sqrt{1+x^6}}{1+2\sqrt{1+x^6}+(1+x^6)-x^6} dx = \int_{0}^{1} \frac{2+2\sqrt{1+x^6}}{2+2\sqrt{1+x^6}} dx = \int_{0}^{1} dx = 1.
\end{aligned}$$