2016 Rasor-Bareis exam solutions

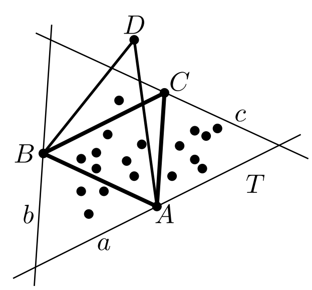
1. *Given a set A of real numbers, we are allowed to replace any two distinct numbers a, b from A by $\frac{a+b}{\sqrt{2}}$ and $\frac{a-b}{\sqrt{2}}$. From initial set $S = \{1, 2, 4\}$, can we apply that operation several times to obtain the set $S' = \{\sqrt{2}, 2\sqrt{2}, 3\}$?*

Solution. No. For a set A, define $s(A)$ to be the sum of the squares of the elements of A, $s(A) = \sum_{a\in A} a^2$. Then $s(A)$ is invariant under our operation, since for any $a, b \in \mathbb{R}$,

$$\left(\frac{a+b}{\sqrt{2}}\right)^2 + \left(\frac{a-b}{\sqrt{2}}\right)^2 = a^2 + b^2.$$

Now, we have $s(S) = 21$ whereas $s(S') = 19$, so S' cannot be obtained from S by several applications of the operation.

2. *There are 2016 points in the plane such that any triangle with the vertices at three of those points has area at most 1. Prove that all these points are contained in a triangle of area 4.*



Solution. Choose three points A, B, C out of these 2016 points for which the triangle $\triangle ABC$ has the maximal area. Construct the straight line a through the point A and parallel to the line BC; the straight line b through the point B and parallel to the line CA; and the straight line c through the point C and parallel to the line AB; the triangle T bounded by the lines a, b, c has area $4\,\text{area}(\triangle ABC) \leq 4$. We claim that all 2016 given points are contained in T. Indeed, if a point D is, say, on the other side of the line c than the points A, B, then the distance from D to AB is larger than the distance from C to AB, so $\text{area}(\triangle ABD) > \text{area}(\triangle ABC)$, which contradicts the choice of $\triangle ABC$.

3. *Let $n \geq 2$ and let $a_1, \dots, a_n > 0$; prove that* $\dfrac{a_1^2}{a_2} + \dfrac{a_2^2}{a_3} + \cdots + \dfrac{a_{n-1}^2}{a_n} \geq 4(a_1 - a_n)$.

Solution. For any $b, c > 0$ we have $b^2 + 4c^2 \geq 4bc$, we have $\frac{b^2}{c} \geq 4(b - c)$. So, for all $i = 1, \dots, n-1$, $\frac{a_i^2}{a_{i+1}} \geq 4(a_i - a_{i+1})$. Summing these up, we get

$$\sum_{i=1}^{n-1} \frac{a_i^2}{a_{i+1}} \geq \sum_{i=1}^{n-1} 4(a_i - a_{j+1}) = 4(a_1 - a_n).$$

4. *Let (a_n) be a bounded increasing sequence of positive real numbers. Prove that*

$$\sum_{n=1}^{\infty}\left(1 - \frac{a_n}{a_{n+1}}\right) < \infty.$$

Solution. This is a series with nonnegative terms, so the comparison criteria apply to it. For any n we have

$$1 - \frac{a_n}{a_{n+1}} = \frac{a_{n+1} - a_n}{a_{n+1}} \leq \frac{a_{n+1} - a_n}{a_1}.$$

Let $a = \lim a_n$. The series $\sum_{n=1}^{\infty} \frac{a_{n+1}-a_n}{a_1}$ is telescoping and converges to $\frac{a-a_1}{a_1}$, so the initial series also converges.

Another solution. Let, again, $a = \lim a_n$. We have $a_n/a_{n+1} \longrightarrow a/a = 1$ as $n \longrightarrow \infty$, so $1 - \dfrac{a_n}{a_{n+1}} \longrightarrow 0$. It is known that $\lim\limits_{x\to 0} \dfrac{\log(1-x)}{x} = -1$, so $\left(\log \dfrac{a_{n+1}}{a_n}\right)\Big/\left(1 - \dfrac{a_n}{a_{n+1}}\right) = \left(-\log \dfrac{a_n}{a_{n+1}}\right)\Big/\left(1 - \dfrac{a_n}{a_{n+1}}\right) \longrightarrow 1$, thus by (the second) comparison principle our series converges iff the series $\sum \log \frac{a_{n+1}}{a_n}$ does. And we have

$$\sum_{n=1}^{\infty} \log \frac{a_{n+1}}{a_n} = \log \prod_{n=1}^{\infty} \frac{a_{n+1}}{a_n} = \log \frac{a}{a_1}.$$

5. *Prove that there are infinitely many positive integers not representable in the form $n^2 + p$, where $n \in \mathbb{N}$ and p is prime.*

Solution. Let us show that inifinitely many (in fact, "almost all", in some sense) perfect squares m^2, $m \in \mathbb{N}$, are not representable this way. Indeed, if $m^2 = n^2 + p$, then $p = m^2 - n^2 = (m-n)(m+n)$, which is impossible unless $m = n+1$ and $p = m+n = 2m-1$. So, m^2 is representable in the form $n^2 + p$ only if $2m-1$ is prime; but there are infinitely many m for which this is not the case.

6. *Let α, β, γ be the angles of a triangle. If $\sin\alpha$, $\sin\beta$, $\sin\gamma$ are all rational, prove that $\cos\alpha$, $\cos\beta$, $\cos\gamma$ are also rational.*

Solution. Let a, b, and c be the length of the sides of the triangles opposite to the angles α, β, and γ respectively. After rescaling the plane, we may assume that $a = 1$. By the sine theorem for triangles, we have $\sin\alpha/a = \sin\beta/b = \sin\gamma/c$, so $b = \sin\beta/\sin\alpha$ and $c = \sin\gamma/\sin\alpha$; since all these sines are rational, we obtain that b and c are rational. But then by the cosine theorem, $\cos\alpha = (b^2 + c^2 - a^2)/(2bc)$, $\cos\beta = (c^2 + a^2 - b^2)/(2ca)$, $\cos\alpha = (a^2 + b^2 - c^2)/(2ab)$ are also rational.