A Collection *Sangaku* Problems

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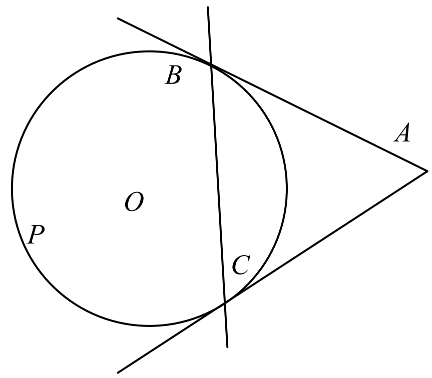
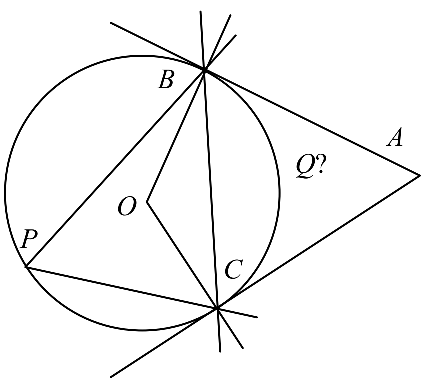
PROBLEM 1: Given two lines tangent to circle O at B and C from a common point A, show that the circle passes through the incenter of triangle ABC.[1]

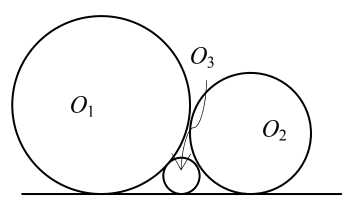




SOLUTION 1 (JMU): Since AB and AC are tangents, each of the base angles of the isosceles triangle ABC measures half of BOC. The sum of the angles between the base and the bisectors of the base angles is therefore also half of BOC. Hence, wherever Q may be, the intersection of the bisectors, may be, BQC is 180° – $BOC/2$.

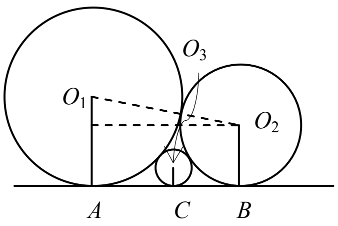
Pick any point P on the arc exterior to the triangle; $BPC = BOC/2$. Since BPC and BQC are supplementary, $BQCP$ must be a cyclic quadrilateral. Since B, C, and P lie on circle O, so must Q, and it is the incenter of ABC. □

PROBLEM 2: What is the relationship between the radii of three circles of different size all tangent to the same line and each externally tangent to the other two?[2]



SOLUTION 2 (F&P): The hypotenuse of the right triangle is

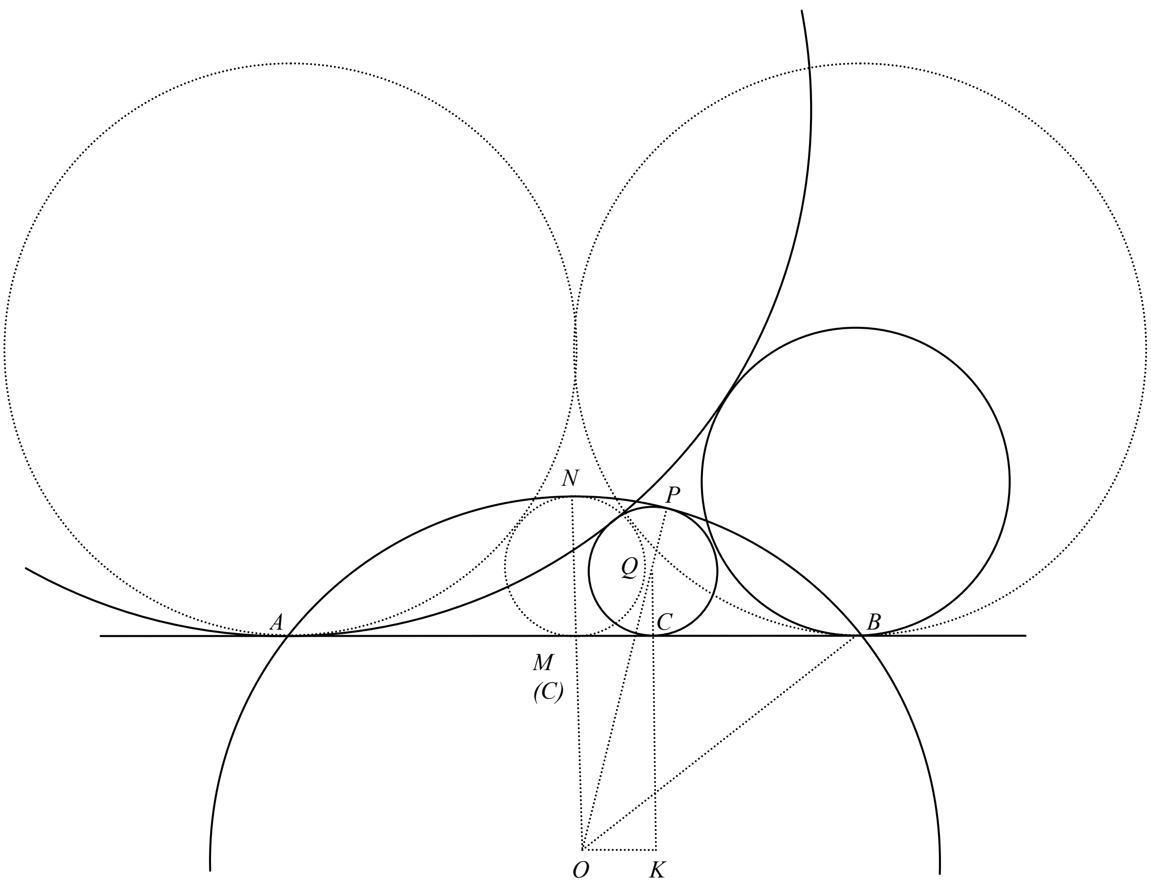
[1] Fukagawa & Pedhoe 1989, 1.1.4; lost tablet from Ibaragi, 1896; no solution given.
[2] Fukagawa & Pedhoe 1989, 1.1.1; well-known; tablet from Gunma, 1824.



$r_1 + r_2$. Its short leg is $r_1 - r_2$, so the other leg is the square root of $(r_1 + r_2)^2 - (r_1 - r_2)^2$. I.e., AB is $2\sqrt{r_1 r_2}$ (twice the geometric mean of the radii).
Likewise, $AC = 2\sqrt{r_1 r_3}$ and $BC = 2\sqrt{r_2 r_3}$. Adding and dividing through by $2\sqrt{r_1 r_2 r_3}$, we obtain

$$\frac{1}{\sqrt{r_3}} = \frac{1}{\sqrt{r_2}} + \frac{1}{\sqrt{r_1}}. \square$$

PROBLEM 3: Suppose two circles tangent to a common line at A and B are tangent to each other externally; that a third circle is tangent to the line and both of them externally; and



that a fourth circle through A and B is tangent to the third circle internally. Show that the fourth circle and the line define a circular segment such that, if the radii of the circles through A and B are varied, the small circle tangent to them remains tangent to both the arc and line of the segment.[3]

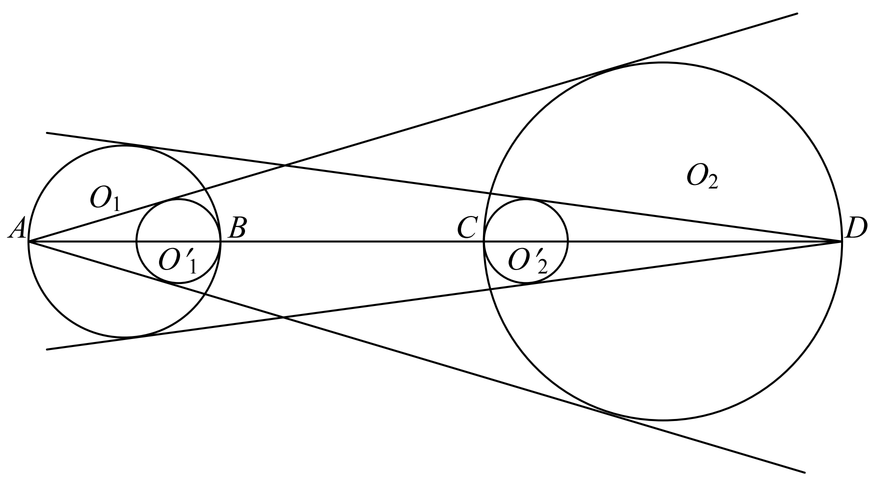
SOLUTION 3 (JMU after A. Bogomolny): Do the easy case first: $M = C$. Then $R_A = R_B = R$, $4r = R$, and $AB = 2R = 8r$. If R_O is the radius of the fourth circle, we have $AM \cdot MB = NM \cdot MO$ or $16r^2 = 2r(2R_O - 2r)$, which leads to $5r = R_O$ or $R_O = 5AB/8$. Since this relationship depends only on the relative locations of A, B, and O, changes in R_A, R_B, and r will not affect it.

Now consider the hard case: $M \neq C$. Writing m for $AM = BM = ½AB$, we have $BO = 5m/4$ and $CO = 3m/4$ because BMO is a 3:4:5 right triangle. Writing x for AC and y for CB, we also have $m = (x + y)/2$ and $MC = (x - y)/2 = KO$. Call the latter n.

We now calculate r two ways: first, using the Pythagorean Theorem in right triangle KOQ, then using the formulae derived in problem 2. On the one hand, we have $(5m/4 - r)^2 - (3m/4 + r)^2 = n^2$, which reduces quickly to $m^2 - 4mr = n^2$. Moreover, $m^2 - n^2 = xy$, so $4mr = xy$. On the other hand, we have $x^2 = 4R_A r$, $y^2 = 4R_B r$, and $(x + y)^2 = 4R_A R_B$. Combining these three equations, we get $xy = 2r(x + y)$. But this is again $xy = 4mr$.

In other words, for any particular C, $r = xy/4m$ is necessary both for circle Q to be inscribed in the circular segment defined by fixed A, B, and O and for all the circles above the line to be tangent as required. □

PROBLEM 4:



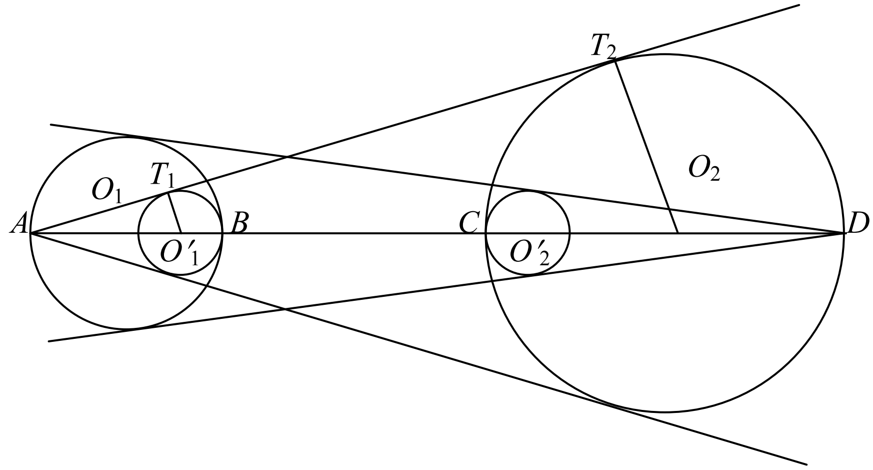
Given two unequal circles with concurrent diameters AB and CD as shown, tangents from A (resp. D) to O_2 (resp. O_1), and circles tangent to B (resp. C) and the two tangents from A (resp. D), prove that the radii of these two circles are equal.[4]

SOLUTION 4 (F&P):

[3] Fukagawa & Pedhoe 1989, 1.1.2; lost tablet from Miyagi, n.d.; the hint $R = \frac{5}{8}AB$ is given, but no solution.

[4] Fukagawa & Pedhoe 1989, 1.3; lost tablet from Aichi, 1842; solution given.

From $\Delta AT_1O_1 \sim \Delta AT_2O_2$, it follows that



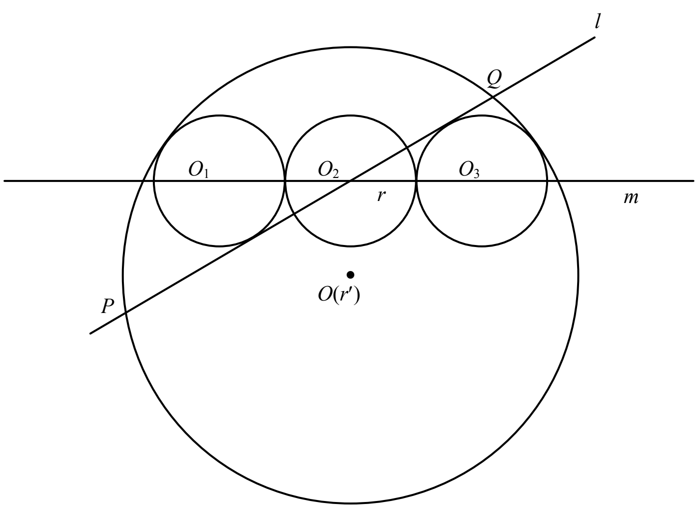
$$\frac{r_1'}{AB - r_1'} = \frac{r_2}{OA_2}$$

and so

$r'_1(AB + BC + CO_2) = r_2(AB - r'_1)$. Hence

$$2r'_1r_1 + r'_1BC + r'_1r_2 = 2r_1r_2 - r_2r'_1$$
$$2r'_1r_1 + r'_1BC + 2r'_1r_2 = 2r_1r_2$$
$$r'_1(2r_1 + BC + 2r_2) = 2r_1r_2$$
$$r_1' = \frac{2r_1r_2}{2r_1 + BC + 2r_2}.$$

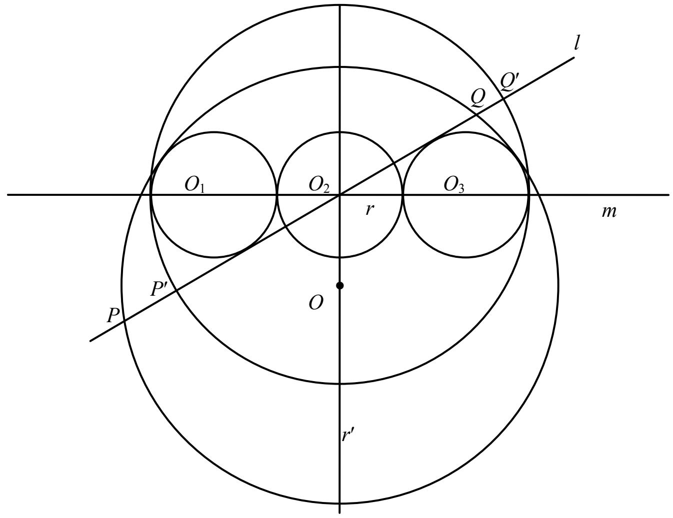
This is algebraically symmetrical: we would have arrived at the same right-side expression for r'_2 if we had started at the other end of the figure. Thus $r'_1 = r'_2$. □



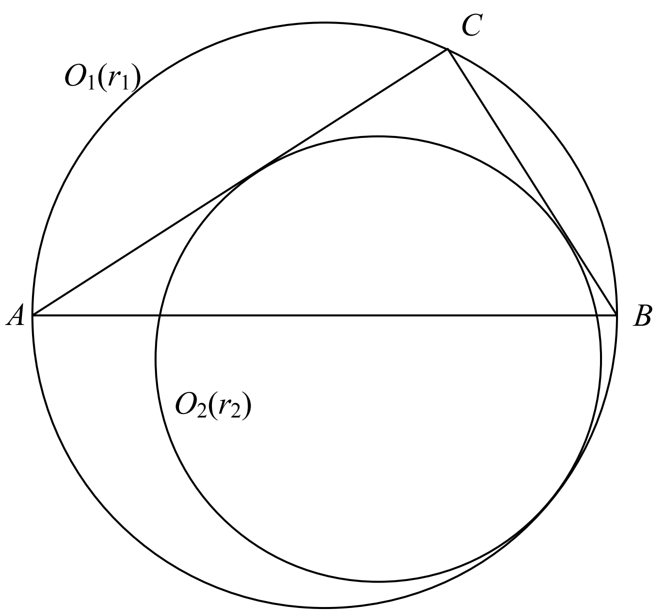
PROBLEM 5: O_1, O_2, and O_3 all have radius r, centers in line m, and form a chain as shown. Line l passes through O_2 and is tangent with O_1 and O_3 on opposite sides of m. Circle $O(r')$ is internally tangent to O_1 and O_3, and is cut by l in P and Q. Prove that $PQ = r' + 3r$.[5]

[5] Fukagawa & Pedhoe 1989, 1.3.3; tablet from Ibaragi, 1871; no solution given.

SOLUTION 5 (JMU): The trick is to superimpose the simplest case on the general case. Start with O coincident with O_2: $P'Q' = 6r = r' + 3r$ is obvious.



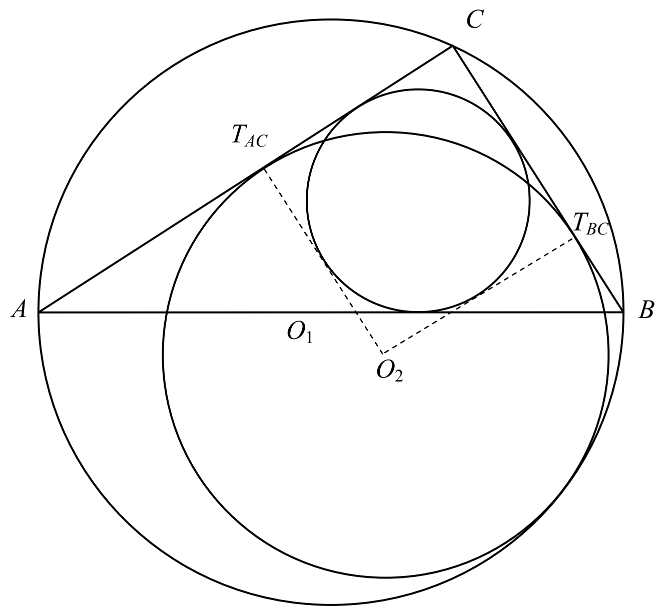
Now, for O below O_2 on the perpendicular to m, PQ is longer than $P'Q'$ by $PP' - QQ'$. That is, $PQ - 6r = PP' - QQ'$. The theorem asserts that $PP' - QQ$ is equal to the change in length of r', which is longer than before by $r' - 3r$. But $PQ - 6r = r' - 3r$ if and only if $PQ = r' + 3r$. □



PROBLEM 6: Given right triangle ΔACB and its circumcircle $O_1(r_1)$, construct circle $O_2(r_2)$ tangent externally to legs a and b and internally to circle O_1. (Such a circle is called mixtilinear.) Prove that $r_2 = a + b - c$.[6]

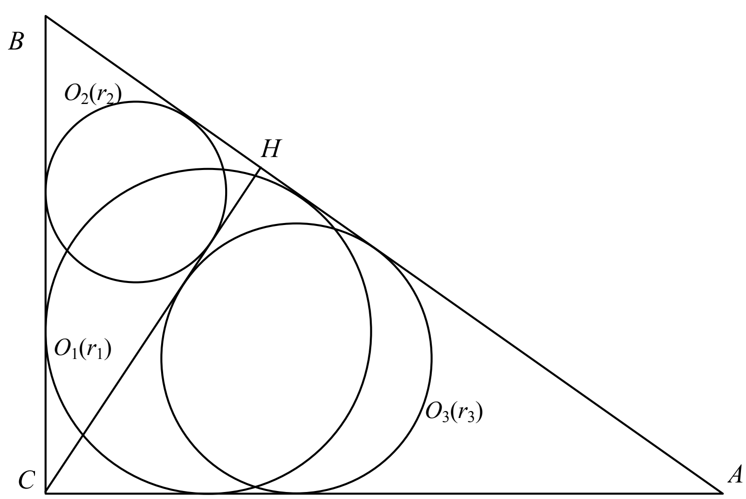
[6] Fukagawa & Pedhoe 1989, 2.2.7; lost tablet from Hyōgo, n.d.; no solution given.

SOLUTION 6 (JMU): The trick is to add the incircle $O_3(r_3)$ and first prove that $r_2 = 2r_3$, which is presented as a separate problem.[7] It is easily done by drawing O_2T_{AC} and O_2T_{BC}, and noting that O_3 touches all four sides of square $CT_{AC}O_2T_{BC}$, each side of which is $2r_3$.



Now in any triangle with semiperimeter s, the distance from C to the point of tangency of the incircle is $s - c$. So in a right triangle such as ACB, $r_3 = s - c = (a + b - c)/2$, which immediately implies $r_2 = a + b - c$. □

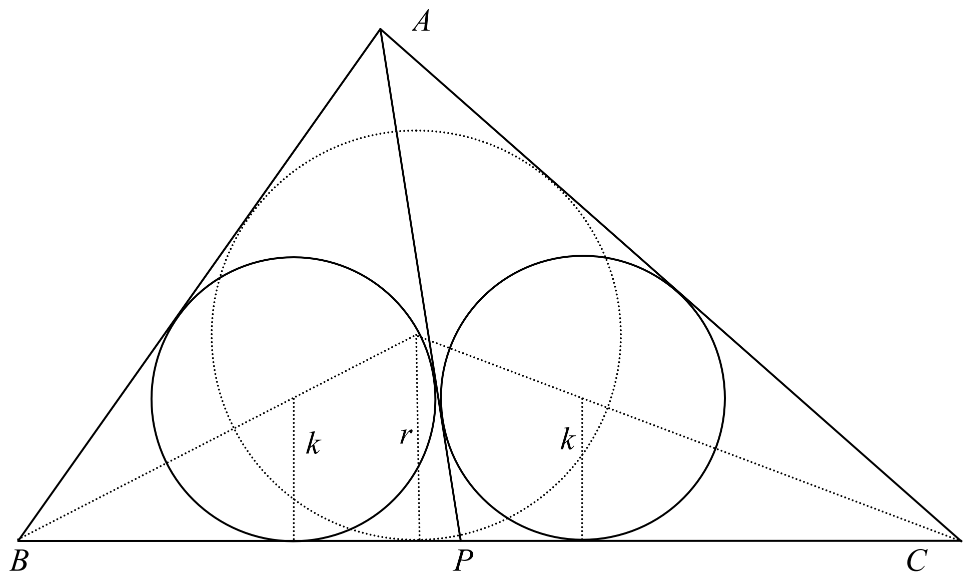
PROBLEM 7: Right triangle ACB is partitioned into two triangles by the altitude CH as shown. Prove that this altitude is the sum of the radii of the three incircles.[8]



SOLUTION 7 (JMU): All three triangles are right. We use the corollary just stated to calculate $2r_1 = a + b - c$, $2r_2 = BH + CH - a$, and $2r_3 = AH + CH - b$. Adding these equations, we get $2r_1 + 2r_2 + 2r_3 = AH + BH + 2CH - c = 2CH$. So $r_1 + r_2 + r_3 = CH$. □

PROBLEM 8: Given two circles of equal radius inscribed as shown below, prove $AP = \sqrt{s(s-a)}$.[9]

[7] Fukagawa & Pedhoe 1989, 2.3; tablet from Iwate, 1842; no solution given.
[8] Fukagawa & Pedhoe 1989, 2.3.2; tablet from Iwate, n.d.; no solution given.



SOLUTION 8 (JMU): In the figure above, r is the inradius of ΔABC, s is its semiperimeter, ΔABP and ΔACP have semiperimeters s_1 and s_2, respectively, but the same inradius k. Using x for AP, observe that $s_1 + s_2 = s + x$ (we will use this fact immediately and once again later). Adding areas, $rs = ks_1 + ks_2$. Hence $x = (rs/k) - s$. By similar triangles,

$$\frac{s-b}{s_1-x} = \frac{r}{k} = \frac{s-c}{s_2-x}.$$

Hence we can calculate x two ways:

$$\frac{s(s-b)}{s_1-x} - s = x = \frac{s(s-c)}{s_2-x} - s$$

Expand each equation, and solve their sum for x:

$$\left.\begin{aligned} s(s-b) - ss_1 + sx &= xs_1 - x^2 \\ s(s-c) - ss_2 + sx &= xs_2 - x^2 \end{aligned}\right\}$$

$$s(2s-b-c) - s(s_1+s_2) + 2sx = x(s_1+s_2) - 2x^2$$

$$sa - s(s+x) + 2sx = x(s+x) - 2x^2$$

$$2x^2 = (s+x)^2 - 2sx - sa$$

$$x^2 = s^2 - sa$$

$$x = \sqrt{s(s-a)}$$

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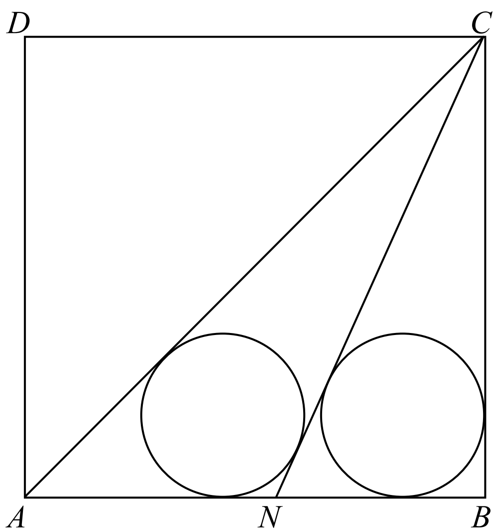
[9] Fukagawa & Pedhoe 1989, 2.2.5; surviving tablet from Chiba, 1897; no solution given.

COROLLARY: If ABC is a right triangle, then $k = \dfrac{ab}{\sqrt{2ab}+a+b+c}$.[10]

PROOF: We have $\Delta = k(b + BP + AP)/2 + k(c + CP + AP)/2 = k(a + b + c + 2AP)/2$, so $k = \dfrac{ab}{a+b+c+2AP}$. In a right triangle, $s - a = r$, so $AP^2 = s(s - a) = rs = \Delta = ab/2$. That is, $4AP^2 = 2ab$ or $2AP = \sqrt{2ab}$. □

PROBLEM 9: $ABCD$ is a square with side a and diagonal AC. The incircles of ACN and BCN are congruent. What is their radius r in terms of a?[11]



SOLUTION 9 (JMU): Because BCN is a right triangle, $r = (BC + BN - CN)/2$ (see problem 6). The congruence of the two incircles implies $CN^2 = s(s - AB)$, where s is the semiperimeter of ABC (proven in problem 8). We know $AC = a\sqrt{2}$, so $s = \dfrac{a\sqrt{2}}{2} + a$.

Hence $CN^2 = \left(\dfrac{a\sqrt{2}}{2} + a\right)\dfrac{a\sqrt{2}}{2} = \dfrac{a^2(\sqrt{2}+1)}{2}$

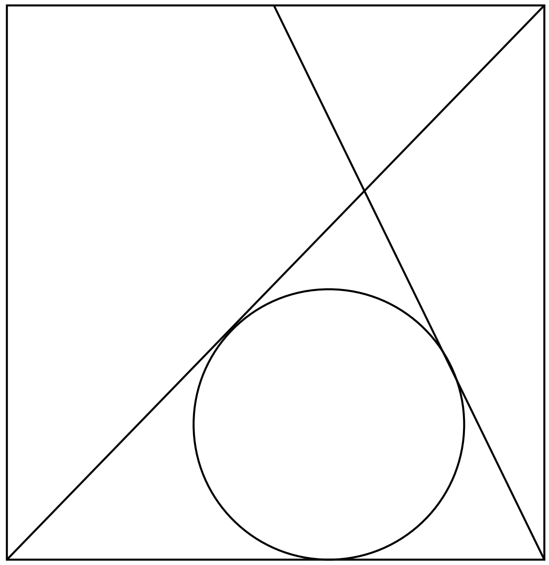
and $CN = \dfrac{a}{2}\sqrt{2+\sqrt{2}}$. Now, since

$BN^2 = \dfrac{a^2(\sqrt{2}+1)}{2} - a^2 = \dfrac{a^2(\sqrt{2}-1)}{2}$,

$BN = \dfrac{a}{2}\sqrt{2-\sqrt{2}}$. So $r = \dfrac{1}{2}\left[a + \dfrac{a}{2}\sqrt{2-\sqrt{2}} - \dfrac{a}{2}\sqrt{2+\sqrt{2}}\right] = \dfrac{a}{2}\left[1 + \dfrac{\sqrt{2-\sqrt{2}}}{2} - \dfrac{\sqrt{2+\sqrt{2}}}{2}\right]$. □

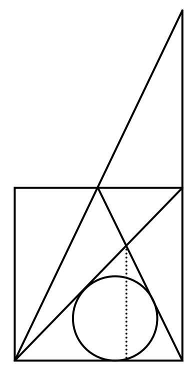
[10] This is Fukagawa & Pedhoe 1989, 2.2.3; lost tablet from Miyagi, 1847; equation given, no solution provided.

[11] Fukagawa & Pedhoe 1989, 3.1.7; surviving tablet from Hyōgo, 1893; the solution given, $r = \frac{1}{2}(1+\sqrt{\sqrt{2}-1})a$, is clearly an error (indeed, $r < \frac{a}{2}$ is obvious from the diagram).



PROBLEM 10: A square with one diagonal is cut by a line from a third vertex to the midpoint of an opposite side. A circle is inscribed in the resulting triangle opposite the midpoint. What is its radius?[12]

SOLUTION 10 (A. Bogomolny): Imagine completing the figure as shown below.



By congruent triangles, it is easy to see that the top of the square bisects the sides of the large right triangle. Hence the two lines within the square are medians of the large right triangle. The apex of the inscribing triangle is its centroid, and divides the two lines within the square in the ratio 1 : 2. For the same reason, if the side of the square is a, the altitude of the inscribing triangle is $\frac{2}{3}a$ (imagine a line parallel to the top and bottom of the square running through the apex of the triangle).

Now the diagonal of the square is $a\sqrt{2}$ and other line in the square is $\frac{a\sqrt{5}}{2}$. The sides of the inscribing triangle are $\frac{2}{3}$ of these lengths, respectively. But in any triangle with base a, altitude thereto h, perimeter p, and inradius r, $2\Delta = pr = ha$. Consequently,

$$\left(a + \frac{2}{3}a\sqrt{2} + \frac{2}{3}\frac{a\sqrt{5}}{2}\right)r = \frac{2a}{3}a$$

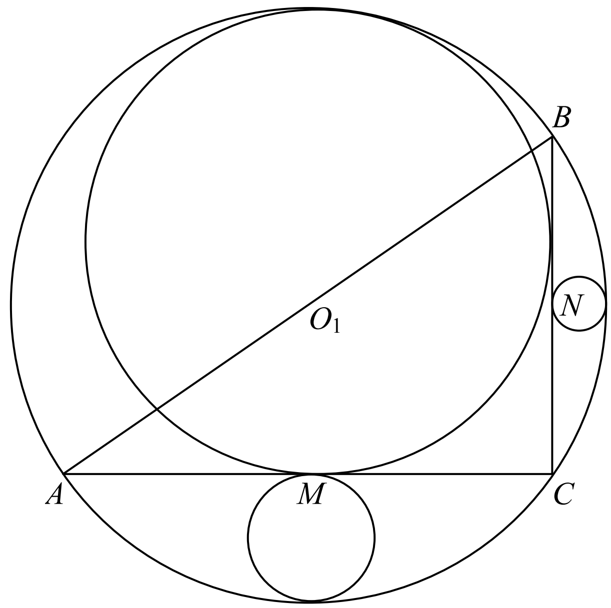
$$(3a + 2a\sqrt{2} + a\sqrt{5})r = 2a^2$$

$$r = \frac{2a}{3 + 2\sqrt{2} + \sqrt{5}}$$

□

[12] Fukagawa & Pedhoe 1989, 3.1.3; surviving tablet from Miyagi, 1877; solution given in the form $r = \frac{2a}{3+\sqrt{5}+\sqrt{8}}$.

PROBLEM 11: A right triangle has three circles tangent to its legs and internally tangent to its circumcircle: O_1 is tangent to both legs; O_2 and O_3 are tangent to legs AC and BC at their midpoints M and N, respectively. Show that $r_1^2 = 32r_2r_3$.[13]



SOLUTION 11 (JMU): The diameters of O_2 and O_3 are the sagittae of chords AC and BC: $v_b = 2r_2$ and $v_a = 2r_3$.

Lemma: In any right triangle, the inradius $r = \sqrt{2v_a v_b}$. Proof:

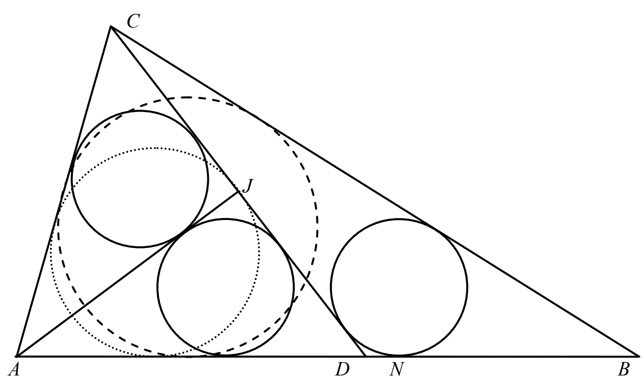
$v_a = 2R - b/2$ $\qquad$ $v_b = 2R - a/2$

$2v_a = c - b$ $\qquad$ $2v_b = c - a$

$$4v_av_b = ab - c(a + b - c)$$
$$4v_av_b = ab - 2cr$$
$$2v_av_b = ab/2 - cr$$
$$2v_av_b = rs - cr = r(s - c) = r^2. \square$$

But $r_1 = 2r$ (problem 6), so $r^2 = r_1^2/4 = 8r_2r_3$. Thus $r_1^2 = 32r_2r_3$. $\square$

PROBLEM 12: In ΔABC, $AB = BC$. If one chooses D on AB and J on CD such that $AJ \perp CD$ and the incircles of ΔACJ, ΔADJ, and ΔBCD all have radius r, then $r = AJ/4$.[14]



SOLUTION 12 (JMU):

The trick to solving the problem expeditiously is to clarify what is given and to prove the converse first.

Notice that the only connection between $AB = BC$ and the geometry of ΔACD is that the incircle of ΔABC (dashes) is tangent to AC at its midpoint. The two incircles of ΔACJ and ΔADJ cannot be tangent to the shared side AJ at the same point, as shown, unless J is the point of tangency of the incircle of ΔACD (dots) with side CD.[15] Hence $AC = AD$. And the two incircles cannot be congruent unless, in addition, either $\angle CAJ = \angle DAJ$ or $AJ \perp CD$, in which case, $\Delta ACJ \cong \Delta ADJ$ and the

[13] Fukagawa & Pedhoe 1989, 2.4.6; surviving tablet from Iwate, 1850; no solution given.

[14] Fukagawa & Rothman 2008:194–96, 212–16

[15] Given as an exercise Honsberger 1995:13, 157.

other condition follows. Therefore, given a fixed isosceles *ABC*, if the congruence of all three small incircles implies $r = AJ/4$, as the problem asserts, then the converse must also be true. Indeed, it is easier to prove.

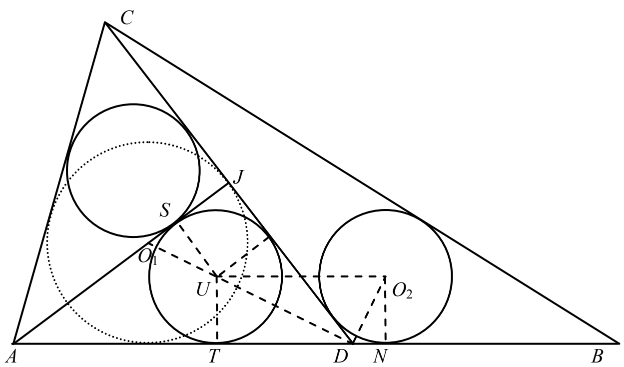
In the figure above, let a, b, d, and h be the lengths of *CD*, $AD = AC$, *DN*, and *AJ*, respectively. Let r_1 and r_2 be the radii of the incircles of ΔACD and ΔBCD, respectively. We use two lemmas, which we prove later:

1. $a - b = 2d$
2. $2r_1r_2 = ad$

Given $h = 4r$, square the equation for the inradius of a right triangle $b = h + a/2 - 2r = 2r + a/2$ to get $b^2 = 4r^2 + 2ar + a^2/4$, and equate this with the Pythagorean result $b^2 = (a/2)^2 + h^2 = a^2/4 + 16r^2$. This yields $a = 6r$, which, with $h = 4r$, implies $b = 5r$. Hence, by Lemma 1, $2d = r$.

Now $\Delta ACD = ha/2 = 12r^2 = r_1s$, where s is the semiperimeter of ΔACD or $8r$. Thus $r_1 = 3r/2 = 3d$. Using Lemma 2, we find $r_2 = a/6 = r$. □

From this, it is clear that solving the original problem comes down to proving, without knowing the value of h, that ΔACJ and ΔADJ are 3:4:5 right triangles.



It's easy to show in the adjoining figure that $\Delta DTU \sim \Delta DO_2U$. This means that *DU* is the mean proportional between *DT* and $O_2U = DT + DN$. In terms of lengths, we therefore have

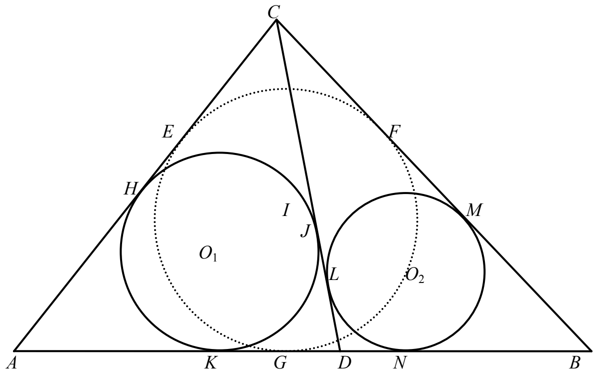
$$\begin{aligned} a/2 - r + d &= [r^2 + (a/2 - r)^2]/(a/2 - r) \\ (a/2 - r)^2 + d(a/2 - r) &= r^2 + (a/2 - r)^2 \\ d(a/2 - r) &= r^2 \\ ad/2 - dr &= r^2 \\ r_1r - dr &= r^2 \\ r_1 - d &= r. \end{aligned}$$

(Lemma 2 takes the form $2r_1r = ad$ since we are assuming $r_2 = r$.)

Since $JS = r$, $O_1S = d$. Thus $JO_1 = r + d = 3d = r_1$.

Hence, by Lemma 2, $2rr_1 = 2(2d)(3d) = 12d^2 = ad$, so $a = 12d = 6r$. Since $a - b = 2d$ (Lemma 1), $b = 10d = 5r$. And because *ADJ* is a right triangle, $a/2 = 3r$ and $b = 5r$ imply $h = 4r$. □

The lemmas follow the general case of any triangle ABC with incircle I. Draw a cevian CD and the incircles O_1 and O_2 of the resulting triangles ACD and BCD. Label the points of tangency of these incircles as shown. Then $EH = GK = DN$.



PROOF: By equal tangents, $CE = CF$ (1), and

$$\begin{array}{llll} AE = AG & BF = BG & CM = CL & DJ = DK \\ AH = AK & BM = BN & CH = CJ & DN = DL. \end{array}$$

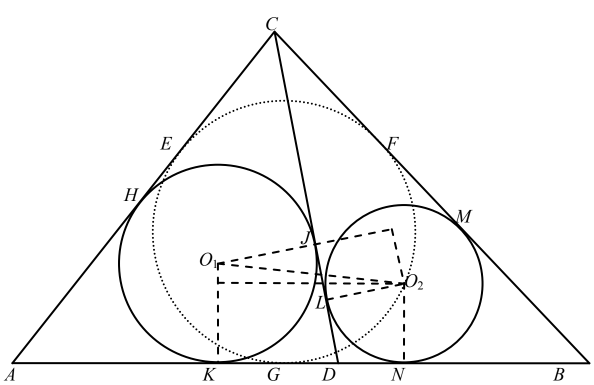
Subtracting the equations in second row from those in the first, we get

$$EH = GK \text{ (2)} \quad FM = GN \text{ (3)} \quad CM - CH = JL \text{ (4)} \quad JL = DK - DN \text{ (5)}$$

Equating the left and right sides of (4) and (5),

$$\begin{array}{rcl} CM + DN & = & DK + CH \\ (CF + FM) + DN & = & DK + (CE + EH) \\ \text{(by 1) } FM + DN & = & DK + EH \\ \text{(by 3) } GN + DN & = & DK + EH \\ (GN - DG) + DN & = & (DK - DG) + EH \\ 2DN & = & GK + EH \\ \text{(by 2) } DN & = & GK \text{ (6).} \end{array}$$

Linking equations (2) and (6), $EH = GK = DN$. □



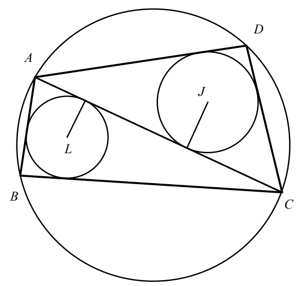
Many more inferences can be drawn from this figure, but we need just two.

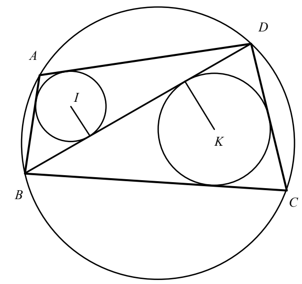
Lemma 1 follows immediately if E and J are the midpoints of AC and CD, respectively. Then $CD/2 - AC/2 = EH = DN$. In the problem, as explained earlier, J must be the midpoint of CD. Requiring that ABC be isosceles forces E to be the midpoint of AC.

Lemma 2 in the general case is $r_1 r_2 = DJ \cdot DL$. The dashed lines in the figure above show that $(r_1 + r_2)^2 + (DJ - DL)^2 = (O_1O_2)^2 = (r_1 - r_2)^2 + KN^2$, but $KN = DK + DN = DJ + DL$

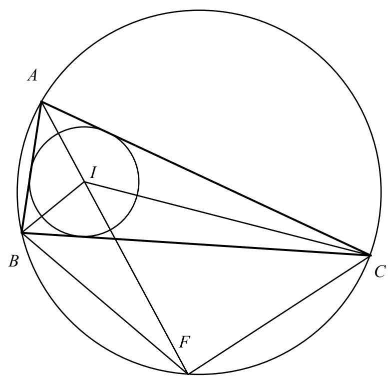
by equal tangents. The rest is just algebra. The traditional solution proves this relation for the problem figure, but it is true even for scalene ΔABC.

PROBLEM 13: Prove that the sums of the radii of the incircles in both triangulations of a (convex) cyclic quadrilateral are equal.[16]





SOLUTION 13 (JMU): There are many ways to prove this theorem. I have put together the following sequence of results on the basis of hints from several different sources.[17]

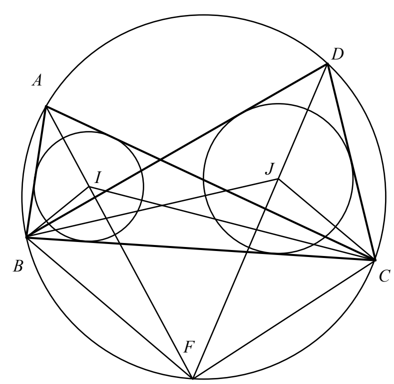


Lemma 1: The bisector from one vertex of a triangle, extended, cuts the circumcircle at the midpoint of the arc subtended by the opposite side of the triangle, which is the center of the circle defined by the other two vertices and the incenter.

γ

Proof: $\angle BIF = \angle BAI + \angle ABI$, that is, half the sum of the vertex angles at A and B. $\angle IBF = \angle CBI + \angle CBF = \angle CBI + \angle CAF$, the same sum. So $\angle BIF = \angle IBF$ and ΔBFI is isosceles. By similar reasoning, so is ΔCFI. Hence $BF = IF = CF$. Moreover, since the $\angle BAF$ and $\angle CAF$, which subtend arcs BF and CF, respectively, are equal, F turns out to be the midpoint of arc BC. □

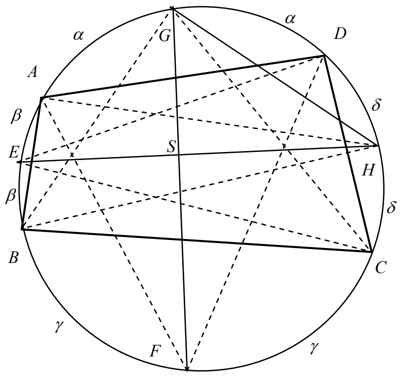
If we add another point D on the circumcircle as shown, it immediately follows that DJ and AI, extended, concur at F and that all four line segments BF, IF, JF, and CF are equal.



Complete the quadrilateral $ABCD$ and construct the eight bisectors that meet at E, F, G, and H, the midpoints of arcs AB, BC, CD, and DA, respectively. (The diagonals of the quadrilateral have been omitted.) It is easy to prove that EH and FG are perpendicular:

[16] Fukagawa & Pedhoe 1989, 3.5(1); lost tablet from Yamagata, 1800.

[17] Most helpful is Ahuja, Uegaki, and Matsushita 2004.



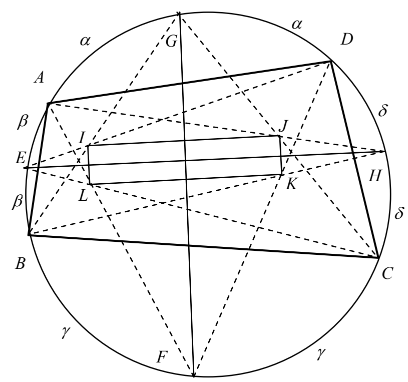
Lemma 2: If a circle is partitioned into four sectors, the lines joining the midpoints of the arcs are perpendicular.

Proof: By hypothesis, $2\pi = 2\alpha + 2\beta + 2\gamma + 2\delta$. Add auxiliary line GH. $\angle GHE = ½(\alpha + \beta)$. $\angle FGH = ½(\gamma + \delta)$. So $\angle GSH = \pi - ½(\alpha + \beta + \gamma + \delta) = \pi - \pi/2 = \pi/2$. □

This leads to the last lemma, which is an impressive theorem in its own right:

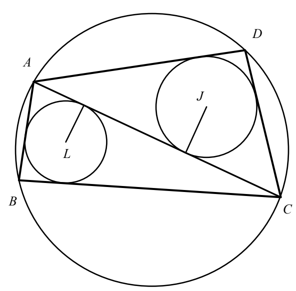
Lemma 3: The incenters of the four triangles formed by the sides of a convex cyclic quadrilateral and its diagonals are the vertices of a rectangle with sides parallel to the lines joining the midpoints of the arcs subtended by the sides of the quadrilateral.

Proof: In the figure, $\angle DEH$ and $\angle HEC$ subtend equal arcs, so EH bisects $\angle DEC$. Lemma 1 assures that $EI = EL$. Thus ΔEIL is isosceles with base IL perpendicular to EH. Applying the same reasoning at H, we conclude that JK is perpendicular to EH, and therefore parallel to JL. Likewise, IJ and LK are parallel and perpendicular to FG. Since EH and FG are themselves perpendicular (Lemma 2), $IJKL$ is a rectangle. □

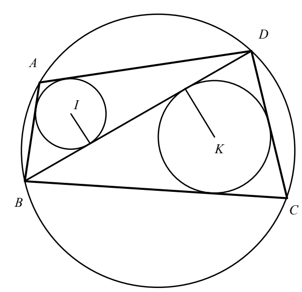


We are now ready to prove the original theorem, which states:

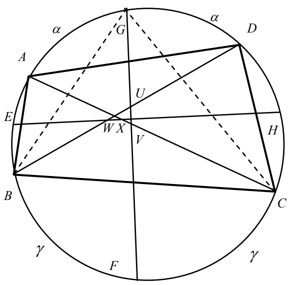
The sums of the radii of the incircles in both triangulations of a (convex) cyclic quadrilateral are the same.



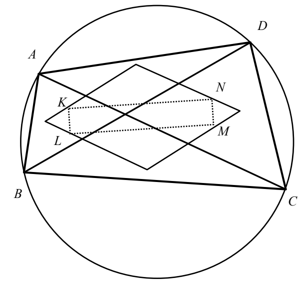
PROOF: If we draw lines through L and J parallel to AC (left) and through I and K parallel to BD (right), the perpendicular distances between each pair of lines will be the sum of the radii of the corresponding pairs of incircles. To prove these sums are equal, it suffices to show that the parallelogram produced by superimposing the two sets of parallel lines is a rhombus, because the two altitudes of a rhombus are equal.



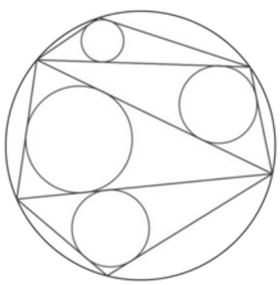
To that end, observe that $\angle ACG = \angle DBG$ because they subtend equal arcs. $\angle BGF = \angle CGF$ for the same reason. Hence, $\Delta BUG \sim \Delta CVG$ with $\angle BUG = \angle CVG$. That is, AC and BD cut GF at the same angle in opposite directions.



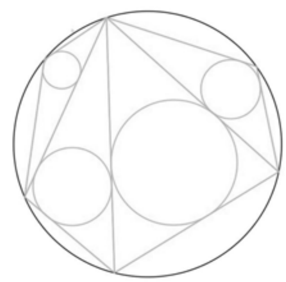
Since EH and FG are perpendicular (Lemma 2), AC and BD likewise cut EH at W and X at the same angle in opposite directions. Hence all lines parallel to the diagonals of the quadrilateral cut the axes of rectangle $KLMN$ (Lemma 3) at the same angles. So the four triangles based on the sides of the rectangle that, together with it, make up the parallelogram, are all isosceles, and we have a rhombus (four sides equal). (Another necessary and sufficient condition for a parallelogram to be a rhombus is that its diagonals be perpendicular: the diagonals of this rhombus lie on EH and FG.) □



COROLLARY: The sums of the inradii in any of triangulation of a (convex) cyclic polygon are all the same.



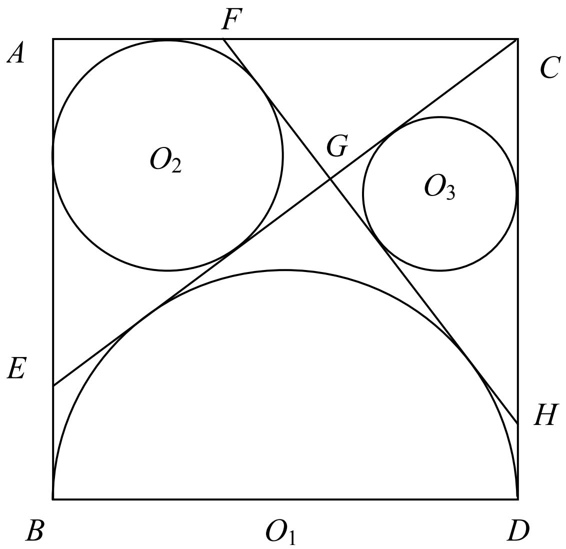
For example, here are two of triaangulations of the same cyclic hexagon. There are many others. Yet the sum of the radii of the incircles is the same for all of them.



PROOF: The previous theorem establishes this theorem for cyclic quadrilaterals. Assume it holds for cyclic n-gons. Every cyclic polygon of $n + 1$ sides can be analyzed as a cyclic n-gon plus a triangle by selecting three adjacent vertices of the starting polygon for the triangle and regarding all the vertices other than the middle one of these three as a cyclic n-gon. Since the same triangle is added to every triangulation of the cyclic n-gon, the theorem holds for the larger polygon too. □

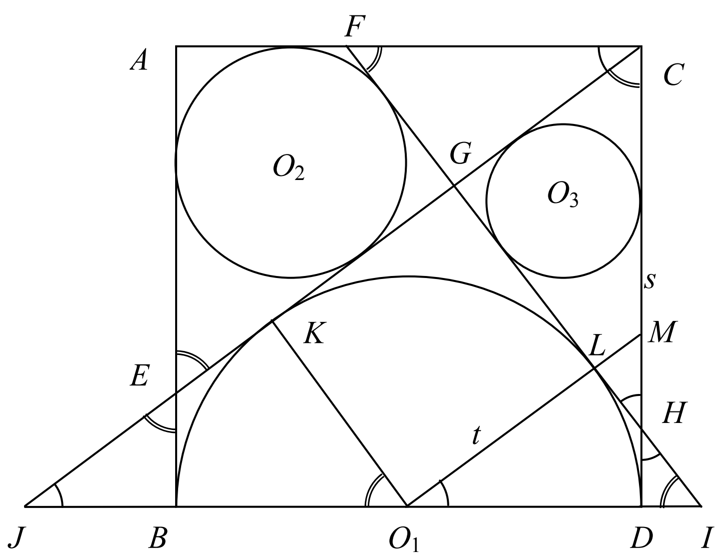
This corollary is frequently described as a theorem by itself.

PROBLEM 14: In square *ABCD*, *CE* is tangent to semicircle BO_1D. *O2* is the incircle of *ACE*. The tangent to O_1 and O_2 meets the sides of the square in *F* and *H* and intersects *CE* in *G*. O_3 is the incircle of *CGH*. Prove that $r_2/r_3 = 3/2$.[18]



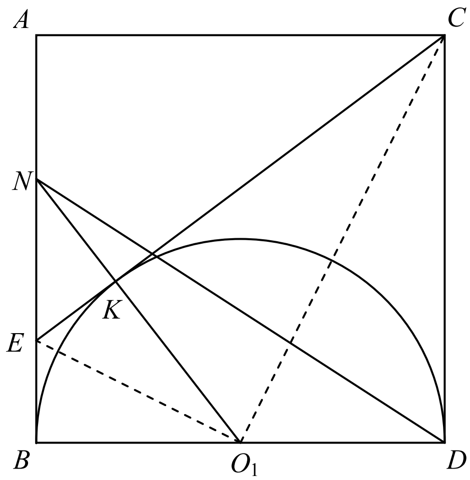
SOLUTION 14 (JMU): First, we prove $CE \perp FH$. Extend *BD*, *CE*, and *FH* and draw the normals KO_1 and LO_1 as shown below. Mark equal angles noting where parallels are cut by transversals, complementary acute angles in known right triangles, vertical angles, and equal angles in similar triangles. There are two kinds of acute angles in each right triangle. Both kinds are found at O_1; since they are complementary, KO_1L must be a right angle. All the right triangles containing both kinds of acute angle are similar, and, by the lemma proved presently, have sides in the ratio 3:4:5.

Let *s* be the side of square *ABCD* and $t = s/2$ be the side of square GLO_1K. Note pairs of tangents from the same points to the same circles: $BE = EK$, $CD = s = CK$, and $DH = HL$. Because of this last pair, if we extend LO_1 to meet *CD* in *M*, $\Delta HLM \cong \Delta HDI$. For later convenience, say that *a*, *b*, and *c* are the lengths of $DI = LM$, $DH = HL$, and $HI = HM$, respectively, noting that $a{:}b{:}c :: 3{:}4{:}5$.



We now prove the key lemma. In the auxiliary figure below, we extend KO_1 to meet *AB* in *N*, and add lines EO_1 and CO_1. EO_1 and CO_1, which form congruent triangles with radii of and equal tangents to circle O_1, bisect supplementary angles, so $\angle CO_1E = 90^\circ$ and KO_1 is the altitude to the hypotenuse of right ΔCO_1E. Hence $KO_1^2 = CK \cdot EK$, or $t^2 = s \cdot EK = 2t \cdot EK$. Therefore $EK = t/2 = BE$. Observe that this implies $AE = \frac{3}{4} AC$, so ΔACE is a 3:4:5 right triangle.

[18] Fukagawa & Pedhoe 1989, 3.2.5, lost tablet of 1838 from Iwate prefecture; no solution given.



Now, returning to the figure above, in ΔACE, $AE + AC - CE = 2r_2 = (s - BE) + s - (s + EK) = s - 2BE = t$ (by the lemma). Thus $r_2 = t/2$. In ΔCGH, $CG + GH - CH = 2r_3 = (s - t) + (t + HL) - (s - DH) = 2b$. Thus $r_3 = b$.

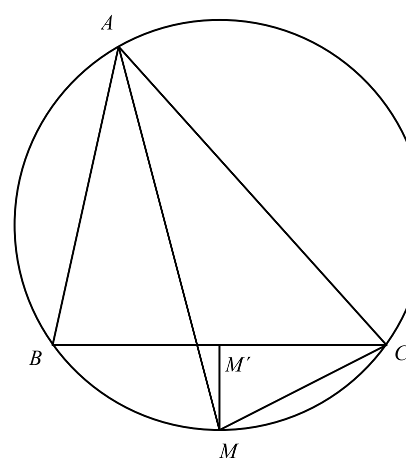
But $c/b = 5/4$, so $b + c = 9b/4$. In ΔDO_1M, $t/(b + c) = 4/3$. Thus $3t = 9b$, or $r_3 = t/3$, $r_2/r_3 = 3/2$. □

PROBLEM 15: In circumscribed triangle ABC, let M' and M be the midpoints of, respectively, chord and arc BC. Then $v_a = M'M$ is the SAGITTA of the chord a. Prove that the square of the distance from a vertex of a triangle to its incenter is four times the product of the sagittae to the adjacent sides.[19]

SOLUTION 15 (F&P):

Lemma: If r is the inradius and s the semiperimeter of triangle ABC, then $(s - b)(s - c) = r^2 + 2v_ar$. (Similar statements hold for the other two sides.)

Proof: We write a' for $s - a$, etc. for convenience.



Square Heron's Formula and divide by s: $r^2s = a'b'c'$ or $a'b'c' = r^2(a' + b' + c')$. Divide by r^3: $(a'/r)(b'/r)(c'/r) = a'/r + b'/r + c'/r$. (Trigonometrically speaking, the product and sum of the cotangents of half of each angle in a triangle are equal.)

AM bisects A because bisectors of angles pass through the midpoints of the circumcircle arcs they subtend. $\angle MAB = \angle BCM$ (both subtend arc BM) = $\angle CAM$, so $a'/r = CM'/M'M = (\tfrac{1}{2}CB)/v_a = (b' + c')/2v_a$. Put this in the equation derived from Heron's Formula:

$$(b' + c')/2v_a + b'/r + c'/r = [(b' + c')/2v_a](b'/r)(c'/r).$$

Multiply through by $2v_ar^2$:

[19] Fukagawa & Pedhoe 1989, 2.2; lost tablet of 1825 from Musashi; solution provided.

$$r^2(b' + c') + 2v_a r(b' + c') = (b' + c')(b'c').$$

Now divide by $(b' + c')$:

$$r^2 + 2v_a r = b'c'.$$

Likewise,

$$r^2 + 2v_b r = a'c'$$
$$r^2 + 2v_c r = a'b'. \ \square$$

To solve the problem, add and multiply the last two equations to obtain

$$2r^2 + 2r(v_b + v_c) = a'(b' + c') \qquad \text{and} \qquad \frac{r^2(r+2v_b)(r+2v_c)}{a'} = a'b'c'.$$

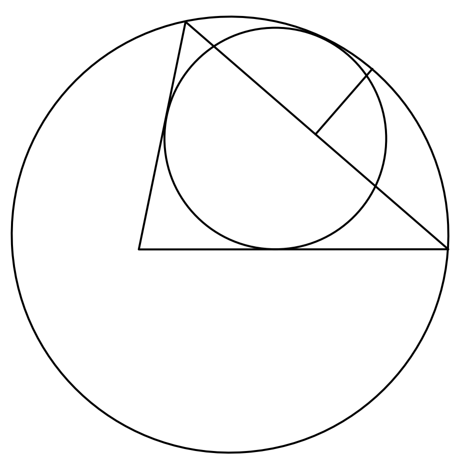
Using these values, the equation $r^2(a' + b' + c') = a'b'c'$ becomes

$$r^2\left[a' + \frac{2r^2 + 2r(v_b + v_c)}{a'}\right] = \frac{r^2(r+2v_b)(r+2v_c)}{a'},$$

which reduces to $a'^2 + r^2 = 4v_b v_c$. But $a'^2 + r^2 = AI^2$. $\square$

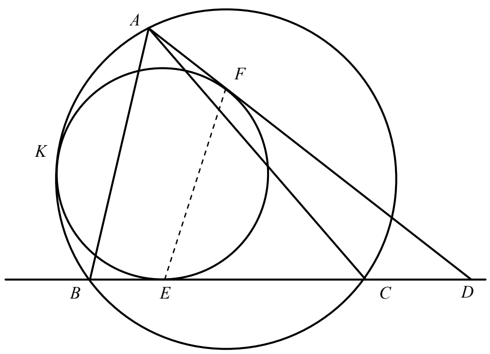
COROLLARY: since $4v_b v_c = AI^2$, $4v_a v_c = BI^2$, and $4v_a v_b = CI^2$, $4^3(v_a v_b v_c)^2 = (AI{\cdot}BI{\cdot}CI)^2$, or $8v_a v_b v_c = AI{\cdot}BI{\cdot}CI$.

PROBLEM 16: Find an expression for the radius of the small circle in the figure at the right in terms of the sides, inradius, and/or sagitta shown of the partially circumscribed triangle.[20]



[20] Fukagawa &Pedhoe 1989, 2.2.8 (1781, n.pl.), "a hard but important problem." A formula for the desired radius is given, but not proved.

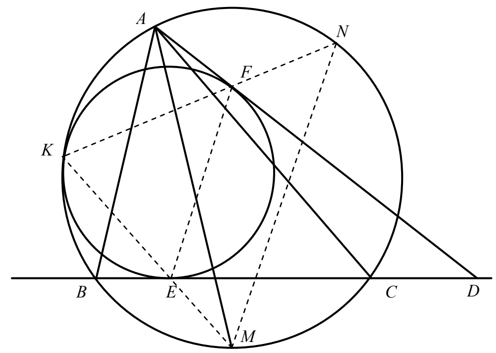
SOLUTION: The last steps are not too hard, but to get to them requires proving a difficult lemma and using some of its implications.[21] The proof presented here is a restatement of a proof by "yetti" posted on MathLinks, 1 January 2005.

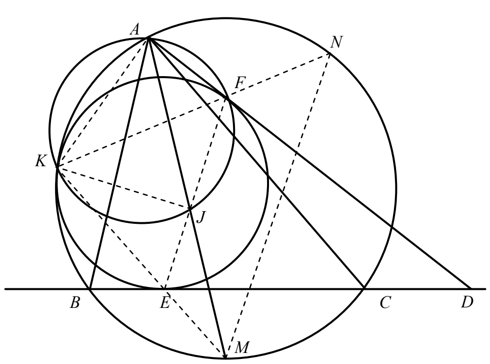


Lemma:

Through vertex A of ΔABC, draw cevian AD with D on BC. Draw circle C_1 tangent to AD at F, CD at E, and the circumcircle C_2 of ΔABC at K. Then the chord EF passes through the incenter I of ΔABC.

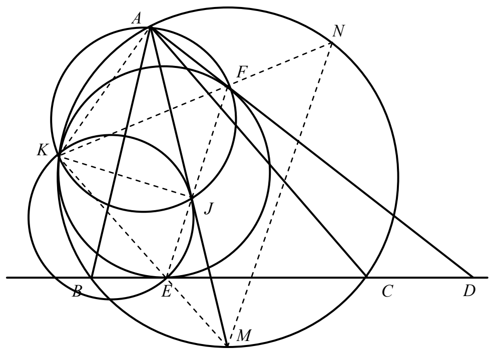
1. Let M and N be the intersections of KE and KF with the circumcircle C_2. Then $MN \parallel EF$ because C_1 is a dilation of C_2 with respect to K. We define M as the image of E under this dilation. Since BC is tangent to C_1 at E, the tangent to C_2 through M must be parallel to BC. But the tangent parallel to the chord subtending an arc touches the arc at its midpoint. Thus M is the midpoint of BC, meaning that AM is the bisector of BAC and hence includes the incenter I.





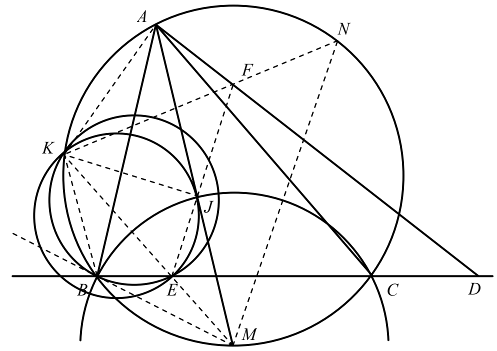
2. Let J be the intersection of AM and EF, and consider the circles passing through A and K. One of them, C_2, contains the chord MN. The corresponding chord for the circle through F must lie on EF because $MN \parallel EF$. And just as F is the image of N with respect to the radical axis AK, J is the image of M. Thus the quadrilateral $AFJK$ is cyclic.

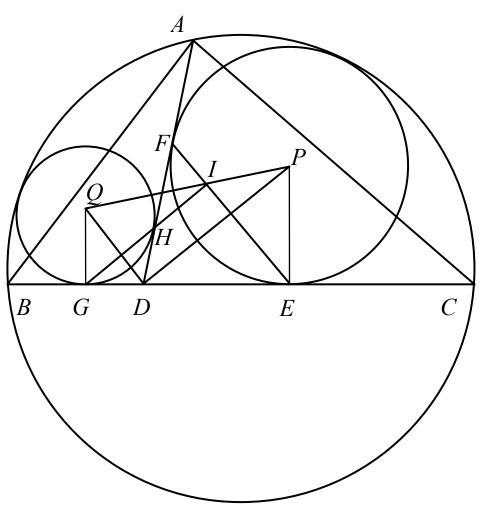
[21] Ayme (2003). Y. Sawayama, an instructor at the Central Military School in Tōkyō published the lemma in 1905 coincidental to solving another problem.



3. We now apply the Miquel Theorem to *AFJ*, selecting *F* on *AF*, *J* on *AJ*, and *E* on *FJ*. The three circles each passing through two of these points and the vertex of *AFJ* not on the line joining them are all concurrent in *K*. Since *E* lies on the extension of *FJ*, it cannot cut *AJ* (i.e. *AM*) twice, so the circumcircle of *KEJ* is tangent to *AM* at *J*.

4. Circle C_3 centered at *M* and radius *BM* passes through *I* because *AM* bisects angle *BAC*. This circle is also orthogonal to the circumcircle of Δ*BKE* because *BKE* = *MAB* = *MAC* = *MBE*, and therefore, by similar triangles, $EK \cdot EM = BM^2$. (Indeed, the circumcircle of Δ*BKE* is its own inverse with respect to C_3.) Since *M* lies on *EK*, all circles with chord *EK* must also be orthogonal to C_3. That includes the circumcircle of *KEJ*. Therefore, *MB* = *MJ*. But *MB* = *MI*, so *I* = *J*. □





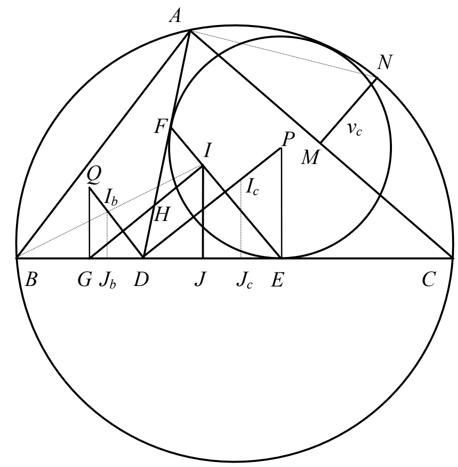
Remark:

Ayme (2003) uses this lemma to solve a famous problem of V. Thébault. The Sawayama-Thébault Theorem states that the centers of the two circles *P* and *Q* and the incircle *I* of triangle *ABC* are collinear. We do not need the theorem itself (the last sentence of the proof) to solve the problem, but we do need the other observations.

To quote Ayme, "According to the hypothesis, $QG \perp BC$, $BC \perp PE$; so $QG \parallel PE$. By [the] Lemma …, *GH* and *EF* pass through *I*. Triangles *DHG* and *QGH* being isosceles …, *DQ* is (1) the perpendicular bisector of *GH*, [and] (2) the *D*-internal angle bisector of triangle *DHG*. *Mutatis mutandis*, *DP* is (1) the perpendicular bisector of *EF*, [and] (2) the *D*-internal angle bisector of triangle *DEF*. As the bisectors of two adjacent and supplementary angles are perpendicular, we have $DQ \perp DP$. Therefore, $GH \parallel DP$ and $DQ \parallel EF$. Conclusion: using the converse of Pappus's theorem … applied to the hexagon *PEIGQDP*, the points *P*, *I* and *Q* are collinear." □

Unfortunately, the labels F and H are reversed in Ayme's paper, and, since transversals cut parallel lines in equal corresponding angles, the invocation of Pappus is rather gilding the lily.

Solution proper: The Sawayama Lemma implies that the incenters of ABC, ACD, and ABD lie on GH and EF. Hence there are pairs of similar right triangles with inradii for one leg. In the figure, we note five of them and the equations they imply:



(1) $DEP \sim DJ_cI_c \quad \Rightarrow \quad x_cr_c = (s_c - b)\cdot DE$
(2) $DJ_cI_c \sim IJE \quad \Rightarrow \quad rr_c = (s_c - b)\cdot EJ$
(3) $DJ_cI_c \sim DJ_bI_b \quad \Rightarrow \quad r_br_c = (s_c - b)(s_b - c)$
(4) $BJI \sim BJ_bI_b \quad \Rightarrow \quad r_b(s - b) = r(s_b - AD)$
(5) $BJI \sim AMN \quad \Rightarrow \quad v_c(s - b) = (b/2)r$

Since AD is common to both ABD and ACB, we also know $s_b + s_c = s + AD$. This allows us to express $s_b - AD$ and DE in (1) and (4) using only terms that occur in the other three equations. Obviously $s_b - AD = s - s_c$, which is easily changed to $(s - b) - (s_c - b)$. From the figure, $DE - EJ = DJ = CD - CJ = CD - (s - c)$. Replacing s with $s_b + s_c - AD$, the last expression becomes $(CD - s_c + AD) - (s_b - c) = (2s_c - b - s_c) - (s_b - c) =$ $(s_c - b) - (s_b - c)$.

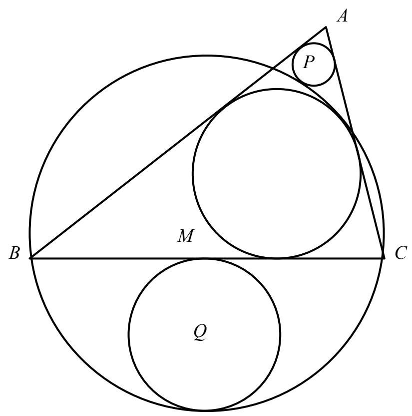
EJ is not so easily eliminated, but dividing (1) by (2), $x_c/r = DE/EJ$. Since as just shown $DE - EJ = (s_c - b) - (s_b - c)$, this means $\dfrac{x_c}{r} - 1 = \dfrac{(s_c - b) - (s_b - c)}{EJ}$. Using (2) once again, we have $x_c - r = \dfrac{(s_c - b)^2 - (s_b - c)(s_c - b)}{r_c}$. So, by (3), $x_c - r = \dfrac{(s_c - b)^2}{r_c} - r_b$. This would be good enough if we weren't restricted to terms involving only ADC, but the problem requires that we eliminate r and r_b from this equation.

We do this with the modified version of (4): $r_b = \dfrac{r[(s - b) - (s_c - b)]}{s - b} = r - \dfrac{r(s_c - b)}{s - b}$, or, by (5), $r - \dfrac{2v_c(s_c - b)}{b}$. Hence $x_c = \dfrac{(s_c - b)^2}{r_c} + \dfrac{2v_c(s_c - b)}{b} = r_c + \dfrac{2v_cr_c^{\,2}}{b(s_c - b)}$. □

This is equivalent to $r_c + \dfrac{2v_c(s_c - AD)(s_c - CD)}{bs_c}$, the solution attached to the original *sangaku* problem, because, squaring Heron's Formula, $r_c^2s_c = (s_c - AD)(s_c - CD)(s_c - b)$.

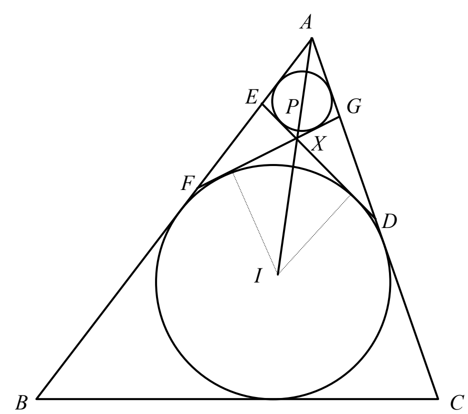
PROBLEM 17: Triangle ABC has incircle (I), to which (O) through B and C is internally tangent. Circle (P) is tangent to AB and AC and externally tangent to (O). Circle (Q) is internally tangent to (O) and tangent to BC at its midpoint M. If r is the inradius of ABC, show that r^2 is 4 times the product of the radii of (P) and (Q).[22]

SOLUTION 17 (JMU): Circle (Q) gives the figure a pleasing balance, but all that matters is its diameter d. We must prove that $r^2 = 2dr_p$, where r_p is the radius of (P).



Our first task is to prove that (O) is tangent to both (I) and (P) if and only if one of their intangents is parallel to BC, which requires some care.

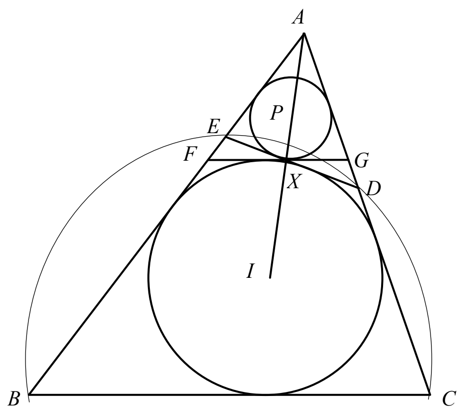
Consider (P) with P on anywhere in the segment AI. Since the two right triangles formed by AI, the two intangents, and radii of (I) are congruent, we see that DE, FG, and AI are concurrent in X, $AXE = AXG$, $AEX \cong AGX$, and $ADE \cong AFG$. The equal vertical angles EXF and GXD have measure $AGF - ADE = AED - AFG$ (an exterior angle of a triangle is the sum of the two opposite interior angles). Hence, Consequently if angles $AFG = ABC$ and $AGF = ACB$, then $FG \parallel BC$ and $ACB - ADE = AED - ABC$. But since $ADE < ACB$ and $AED > ABC$, this equation holds for the figure only if $ADE = ABC$ and $AED = ACB$. This proves



Lemma: FG (DE) is parallel to BC if and only if DE (FG) is its antiparallel in triangle ABC.

Now let M be the midpoint of BC, and consider the coaxial system Γ of circles with centers on the perpendicular bisector of BC. The typical circle cuts AB and AC in two points that, together with B and C, form a cyclic quadrilateral. If $FG \parallel BC$, then $BCDE$ is one of these because, in any cyclic quadrilateral, an exterior angle equals to interior angle at the non-adjacent vertex, and we have just shown that $ADE = ABC$ and $AEG = ACB$.

[22] Fukagawa & Pedhoe 1989, 2.4.2.

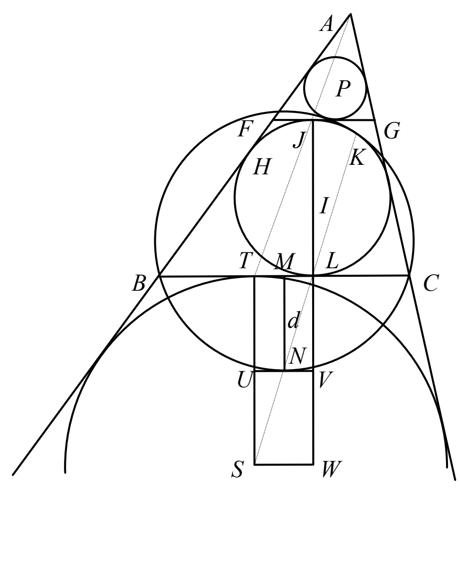


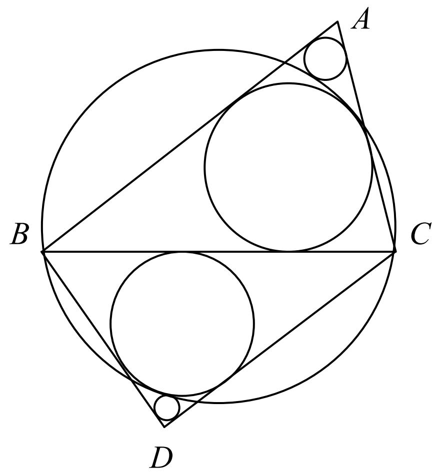
DE touches (P) and (I) in two distinct points. Because all chords cut off by circles in Γ are parallel to DE, the circle (O) in Γ that passes through D or E must also pass through the other. So by the lemma, there is a circle (O) through B and C tangent to (I) internally and to (P) externally, as specified in the problem, if and only if one of their intangents (viz. FG) is parallel to BC.

Since $AFG \sim ABC$, the rest is easy: let H be the point on AB touched by (I). $AH = s - a$, where s is the semiperimeter of ABC. Since (I) is the excircle of AFG, AH is also the semiperimeter of AFG. Hence $r_p/r = (s - a)/s$. Therefore $2dr_p = 2dr(s - a)/s$. If $2dr(s - a)/s = r^2$, then $2d(s - a) = rs$ = area ABC. This is true if and only if $2d$ is the radius of the excircle touching BC.

Since excircle (S) belongs to Σ, point T, where AJ cuts BC, is its contact point on BC. AI is concurrent with the bisectors of the exterior angles at B and C in S, and $ST \perp BC$ just as $LJ \perp BC$. Therefore, drawing parallels to BC through N and S, we get rectangles $LTUV$, $LTSW$, and $SUVW$. $MN = d = TU = LV$ but we don't yet know the length of $US = VW$. However, $BT = LC = s - c$; therefore $MT = ML$. Hence $LMN \cong NUS$: N lies on the diagonal of $LTSW$, and $US = VW = d$. □

Incidentally, K, L, and N are collinear because (I) is inscribed in the circular segment of (O) bounded by arc BKC and chord BC.





COROLLARY[23]: In the case of two triangles, ABC and BCD, if the radii of the two circles tangent to BC are r_1 and r_2 and the radii of the two small circles at A and D are r_1' and r_2', then $r_1r_2 = (r_1'r_2'/2BC)^2$. This is the result stated for Fukagawa & Pedhoe's problem 2.5.5. For if the diameter perpendicular to BC measures d_1 above BC and d_2 below, $2d_1r_1 = r_1'^2$ and $2d_2r_2 = r_2'^2$. Multiply these equations together, noting that $d_1d_2 = (BC/2)^2$. □

[23] Fukagawa & Pedhoe 1989, 2.5.5.

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