Sphere Packings

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Abstract. The density of a (point) lattice sphere packing in n dimensions is the volume of a sphere in $\mathbb{R}^n$ divided by the volume of a fundamental region of the (point) lattice. We will give examples of packings where the centers of the spheres are points on the $\mathbb{Z}^n$, A_n, and D_n lattices, calculate their densities, center densities, and covering radii, and state the densest known lattice packings in 1 through 4 dimensions. We will then explain how the $\mathbb{Z}^4$ and D_8 packings have enough room for a second copy of the respective packing to be placed next to the first one without any sphere intersections, resulting in lattice packings with twice the densities of the originals. In addition, we will mention Rogers's upper bound and Minkowski's lower bound regarding sphere packing densities, and also prove Mordell's inequality.

1 Sphere packings

Note. Unless otherwise noted, information is from [Conway and Sloane]. All decimal numbers will be rounded to 4 decimal places.

1.1 Definitions and notes

The volume of a unit n-dimensional sphere will be denoted by V_n.

Definition.

- A (point) **lattice** in $\mathbb{R}^n$ is a discrete subgroup of $\mathbb{R}^n$ under addition containing the origin [MathWorld, "Point lattice"].
- A **fundamental region** of L is a subset E of $\mathbb{R}^n$ that satisfies $E + L = \mathbb{R}^n$ and $(E + l_1) \cap (E + l_2) = \emptyset$ for any $l_1, l_2 \in L$, $l_1 \neq l_2$.

  Example. For a lattice L in $\mathbb{R}^n$, there exist n vectors $\begin{matrix} \mathbf{v}_1 = (v_{11}, \ldots, v_{1m}) \\ \vdots \\ \mathbf{v}_n = (v_{n1}, \ldots, v_{nm}) \end{matrix}$ in $\mathbb{R}^m$, where $m \geq n$, such that for any $\mathbf{x} \in L$, $\mathbf{x} = \sum_{i=1}^{n} c_i \mathbf{v}_i$ where $c_i \in \mathbb{Z}$. The region $\{a_1\mathbf{v}_1 + \cdots + a_n\mathbf{v}_n : 0 \leq a_i < 1\}$. In this talk the only fundamental regions that will be discussed are of this type.

- $\mathbf{M} = \begin{pmatrix} \mathbf{v}_1 \\ \vdots \\ \mathbf{v}_n \end{pmatrix}$ is called the **generator matrix** for L.
- The **Gram matrix A** is defined as $\mathbf{A} = \mathbf{M}\mathbf{M}^{\mathsf{T}}$.
- The **determinant** of a lattice L is $\det L = \det \mathbf{A}$.

  If $\mathbf{M}$ is square, $\det L = (\det \mathbf{M})^2$. $\det L$ can be considered to be *square* of the volume of the fundamental region, regardless of the particular region used.

- The **density** Δ of a lattice sphere packing is

$$\boxed{\Delta = \frac{V_n r^n}{\text{Volume of the fundamental region}}},$$

where r is the radius of each sphere.

In other words, $\Delta = \frac{V_n r^n}{\sqrt{\det L}}$. For a lattice with Gram matrix $\mathbf{A}$, the definition can be rewritten as $\Delta = \frac{V_n r^n}{\sqrt{\det \mathbf{A}}}$. We also define Δ_n to be the maximal density of any lattice sphere packing in $\mathbb{R}^n$.

- The **center density** δ of a sphere packing is

$$\boxed{\delta = \frac{\Delta}{V_n}},$$

In other words, $\delta = \frac{r^n}{\sqrt{\det \mathbf{A}}}$ for lattices. We also define δ_n to be the maximal center density of any lattice sphere packing in $\mathbb{R}^n$.

- A **deep hole** of a lattice L is a point $\mathbf{x} \in \mathbb{R}^n$ such that $\operatorname{dist}(\mathbf{x}, L) = \max_{\mathbf{y} \in \mathbb{R}^n} \{\mathbf{y}, L\}$.

- The **covering radius** R of a lattice L is half the distance between a point of L and its nearest deep hole.

1.2 Various lattices

"$\cong$" denotes the equivalence of two lattices. Two lattices are **equivalent** if one can be transformed into the other using scaling, rotations, and/or reflections. More precisely, L and L' are two equivalent lattices if and only if their generator matrices $\mathbf{M}$ and $\mathbf{M}'$ satisfy $\mathbf{M}' = c\mathbf{UMB}$, where c is a nonzero constant, $\mathbf{U}$ has integer entries and $\det \mathbf{U} \in \{-1, 1\}$, and $\mathbf{B}$ is an orthogonal matrix with real entries.

1.2.1 The cubic lattices $\mathbb{Z}^n$

Lattice	$\mathbb{Z}^n$									
Definition	$\{(x_1, \ldots, x_n) \in \mathbb{Z}^n\},\ n \geq 1$									
Basis vectors	$\mathbf{v}_1 = (1, 0, 0, \ldots, 0, 0)$ $\mathbf{v}_2 = (0, 1, 0, \ldots, 0, 0)$ $\vdots$ $\mathbf{v}_n = (0, 0, 0, \ldots, 0, 1)$									
Generator matrix	$\mathbf{M}_{\mathbb{Z}^n} = \begin{pmatrix} 1 & & \\ & \ddots & \\ & & 1 \end{pmatrix}_{n \times n}$									
Gram matrix	$\mathbf{A}_{\mathbb{Z}^n} = \begin{pmatrix} 1 & & \\ & \ddots & \\ & & 1 \end{pmatrix}_{n \times n}$									
Volume of fundamental region	$\sqrt{\det \mathbf{A}_{\mathbb{Z}^n}} = 1$									
Radius of sphere	$r = \frac{1}{2}$									
Center density	$\delta = \left(\frac{1}{2}\right)^n$									
Covering radius	$R = \frac{1}{2}\sqrt{n}$									
Dimension	**1**	**2**	**3**	**4**	**5**	**6**	**7**	**8**	$\cdots$	**24**
Density Δ	1	0.7854	0.5236	0.3084	0.1645	0.0807	0.0369	0.0159	$\cdots$	$1.1501 \cdot 10^{-10}$

Densities calculated from [MathWorld, "Hypersphere"].

Sphere packings from the $\mathbb{Z}^2$ and $\mathbb{Z}^3$ lattices can be seen in **[Figure 1]** and **[Figure 3]**.

In n dimensions, consider a cube of side length 2 surrounding a cube of spheres (each of radius $\frac{1}{2}$) such that the cube just touches all the spheres. Draw a sphere centered at the center of the cube with the largest possible radius such that it does not intersect the other spheres. The radius of this center sphere is $r_{n,\text{center}} = \frac{1}{2}\left(2\sqrt{n} - 2\left(\frac{1}{2}\sqrt{n} + \frac{1}{2}\right)\right) = \frac{1}{2}\sqrt{n} - \frac{1}{2}$. (The covering radius is simply this value plus the radius of a lattice sphere.) For $n = 2$ and $n = 3$, one gets $r_{2,\text{center}} \approx 0.2071$ and $r_{3,\text{center}} \approx 0.3660$, both less than 0.5. However, $r_{4,\text{center}} = 0.5$ and $r_{9,\text{center}} = 1$, which means that in 9 dimensions, the center sphere will touch the sides of the surrounding cube! Since $\{r_{n,\text{center}}\}_{n \in \mathbb{N}}$ is monotone increasing and unbounded as $n \to \infty$, $r_{n,\text{center}} > 1$ in dimensions 10 and above, which results in parts of the sphere protruding from the surrounding cube. The specific case $n = 4$ will be discussed later.

1.2.2 The lattices A_n

Lattice	A_n								
Definition	$\{(x_0, x_1, \dots, x_n) \in \mathbb{Z}^{n+1} : x_0 + x_1 + \cdots + x_n = 0\}$, $n \geq 1$								
Basis vectors	$\mathbf{v}_1 = (-1, 1, 0, 0, \dots, 0, 0, 0)$ $\mathbf{v}_2 = (0, -1, 1, 0, \dots, 0, 0, 0)$ $\vdots$ $\mathbf{v}_n = (0, 0, 0, 0, \dots, 0, -1, 1)$								
Generator matrix	$\mathbf{M}_{A_n} = \begin{pmatrix} -1 & 1 & & & \\ & -1 & 1 & & \\ & & \ddots & \ddots & \\ & & & -1 & 1 \end{pmatrix}_{n\times(n+1)}$								
Gram matrix	$\mathbf{A}_{A_n} = \begin{pmatrix} 2 & -1 & & & \\ -1 & 2 & -1 & & \\ & -1 & \ddots & \ddots & \\ & & \ddots & 2 & -1 \\ & & & -1 & 2 \end{pmatrix}_{n\times n}$								
Volume of fundamental region	$\sqrt{\det \mathbf{A}_{A_n}} = \sqrt{n+1}$								
Radius of sphere	$r = \frac{1}{\sqrt{2}}$								
Center density	$\delta = \frac{\left(\frac{1}{\sqrt{2}}\right)^n}{\sqrt{n+1}}$								
Covering radius	$R = \frac{1}{\sqrt{2}}\sqrt{\frac{2\lfloor\frac{n+1}{2}\rfloor\left(n+1-\lfloor\frac{n+1}{2}\rfloor\right)}{n+1}}$								
Dimension	**1**	**2**	**3**	**4**	**5**	**6**	**7**	**8**	$\cdots$	**24**
Density Δ	1	0.9069	0.7405	0.5517	0.3799	0.2442	0.1476	0.0846	$\cdots$	$9.4217 \cdot 10^{-8}$

A_2 is the hexagonal lattice which gives the densest possible circle packing **[Figure 2]** and **[Figure 4]**.

1.2.3 The checkerboard lattices D_n

Lattice	$\boldsymbol{D_n}$
Definition	$\{(x_1,\ldots,x_n)\in\mathbb{Z}^n : x_1+\cdots+x_n \text{ is even}\}$, $n\geq 3$
Basis vectors	$\mathbf{v}_1=(-1,-1,0,0,\ldots,0,0,0)$ $\mathbf{v}_2=(1,-1,0,0,\ldots,0,0,0)$ $\mathbf{v}_3=(0,1,-1,0,\ldots,0,0,0)$ $\vdots$ $\mathbf{v}_n=(0,0,0,0,\ldots,0,1,-1)$
Generator matrix	$\mathbf{M}_{D_n}=\begin{pmatrix}-1 & -1 & & & & \\ 1 & -1 & & & & \\ & 1 & -1 & & & \\ & & \ddots & \ddots & & \\ & & & 1 & -1 \end{pmatrix}_{n\times n}$
Gram matrix	$\mathbf{A}_{D_n}=\begin{pmatrix}2 & 0 & -1 & & & & \\ 0 & 2 & -1 & & & & \\ -1 & -1 & 2 & -1 & & & \\ & & -1 & 2 & -1 & & \\ & & & -1 & 2 & \ddots & \\ & & & & \ddots & \ddots & -1 \\ & & & & & -1 & 2\end{pmatrix}_{n\times n}$
Volume of fundamental region	$\sqrt{\det \mathbf{A}_{D_n}}=2$
Radius of sphere	$\frac{1}{\sqrt{2}}$
Center density $\boldsymbol{\delta}$	$\frac{\left(\frac{1}{\sqrt{2}}\right)^n}{2}$
Covering radius	$R=\begin{cases}1 & n=3\\ \frac{1}{\sqrt{2}}\sqrt{\frac{n}{2}} & n\geq 4\end{cases}$

Dimension	**1**	**2**	**3**	**4**	**5**	**6**	**7**	**8**	$\cdots$	**24**
Density $\boldsymbol{\Delta}$	—	—	0.7405	0.6169	0.4653	0.3230	0.2088	0.1268	$\cdots$	$2.3554\cdot 10^{-7}$

The D_3 lattice is known as the face-centered cubic lattice. It can be seen in **[Figure 5]**.

As discussed earlier, the covering radius of $\mathbb{Z}^4$ is 1, which allows for another copy of $\mathbb{Z}^4$ to fit in the deep holes in the lattice without modifying the existing lattice or the sphere radius. In precise terms, this configuration is the set $\left(\mathbb{Z}^4\right)^+ := \mathbb{Z}^4\cup\left(\mathbb{Z}^4+\left(\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{1}{2}\right)\right)$ [Weaire and Aste]. Since $\left(\mathbb{Z}^4\right)^+$ consists of two copies of $\mathbb{Z}^4$ in the same space, it has double the density of $\mathbb{Z}^4$, in other words 0.6169. $\left(\mathbb{Z}^4\right)^+$ is in fact equivalent to D_4. It is also true that $D_4^+ := D_4\cup\left(D_4+\left(\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{1}{2}\right)\right)\cong\mathbb{Z}^4$, although this time the sphere radius changes. In addition, $D_n^+ := D_n\cup\left(D_n+\left(\frac{1}{2},\ldots,\frac{1}{2}\right)\right)$ is a lattice iff n is even. Note that when $n=8$, $R=\sqrt{2}$, so one can fit in another copy of D_8 into the deep holes of the existing copy of D_8 without modifying it. This lattice D_8^+ is known as the E_8 lattice.

Theorem. *(Gauss) D_3 is the densest lattice packing in 3 dimensions. This lattice is unique up to reflections and rotations.*

Proof. See [Zong]. $\square$

1.3 Some theorems

1.3.1 Sphere packing density bounds

Theorem. *(Rogers's upper bound) Consider a regular n-dimensional simplex of side length 2 in $\mathbb{R}^n$. Draw $n+1$ n-dimensional spheres of unit radius, each centered at a vertex of the simplex. Let σ_n be the proportion of the simplex that the interiors of the spheres fill. Then the density of any sphere packing in $\mathbb{R}^n$ satisfies*

$$\boxed{\Delta \leq \sigma_n}.$$

This bound coincides with the hexagonal packing in $n = 2$.

Theorem. *(Minkowski) In $\mathbb{R}^n$, lattices of density*

$$\boxed{\Delta \geq \frac{\zeta(n)}{2^{n-1}}}$$

exist, where $\zeta(n)$ is the Riemann zeta function, $\zeta(n) = \sum_{k=1}^{\infty} \frac{1}{k^n}$.

The general form of this theorem, valid for centrally symmetric convex bodies, is called the *Minkowski-Hlawka Theorem* [Zong]. The best known packing densities in some dimensions are shown in the following table.

Dimension	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**···**	**24**
Upper bound (Rogers)	0.9069	0.7796	0.6478	0.5257	0.4192	0.3298	0.2568	···	0.0025
Best known lattice packing	A_2 0.9069	$D_3 \cong A_3$ 0.7405	D_4 0.6169	$\Lambda_5 \cong D_5$ 0.4653	0.3730	0.2953	0.2537	···	Leech 0.0019
Lower bound (Minkowski)	0.8220	0.3005	0.1353	0.0648	0.0318	0.0158	0.0078	···	$1.1921 \cdot 10^{-7}$
$\mathbb{Z}^{\boldsymbol{n}}$	0.7854	0.5253	0.3084	0.1645	0.0807	0.0369	0.0159	···	$1.1501 \cdot 10^{-10}$

The densest known lattice packing in 24 dimensions is called the Leech lattice. It is 10 *million* times as dense as $\mathbb{Z}^{24}$! Lastly, we have an inequality that places an upper bound on the density of a lattice in $\mathbb{R}^n$ given the highest possible density of any lattice in $\mathbb{R}^{n-1}$.

Definition. For a lattice L, the **minimal norm** μ of L is

$$\boxed{\mu = \min_{\mathbf{x} \in L, \mathbf{x} \neq \mathbf{0}} \{\mathbf{x} \cdot \mathbf{x}\}}.$$

Claim. For a lattice L, the following equation holds: $\delta = \left(\frac{\mu}{4}\right)^{\frac{n}{2}} \frac{1}{\sqrt{\det L}}$.

Proof. Since the radius of the sphere is equal to half the distance between the origin and the closest nonzero point in L, $\delta = \frac{r^n}{\sqrt{\det L}} = \frac{\left(\frac{\sqrt{\mu}}{2}\right)^n}{\sqrt{\det L}} = \left(\frac{\mu}{4}\right)^{\frac{n}{2}} \frac{1}{\sqrt{\det L}}$. ☐

Definition. For a lattice L, the **dual lattice** L^* is defined as

$$\boxed{L^* = \{\mathbf{x} \in \mathbb{R}^n : \mathbf{x} \cdot \mathbf{y} \in \mathbb{Z} \text{ for all } \mathbf{y} \in \Lambda\}}.$$

Since the Gram matrix of L^* is the inverse of the Gram matrix of L, $\det L^* = (\det L)^{-1}$.

Theorem. *Let L_n be a lattice in $\mathbb{R}^n$ (not necessarily integral), and let S be a k-dimensional subspace of $\mathbb{R}^n$. Let $E_k = L_n \cap S$ and $F_{n-k} = L_n^* \cap S^{\perp}$. Then $\det F = \frac{\det E}{\det L_n}$.*

Proof. Let $\{\mathbf{e}_1, \ldots, \mathbf{e}_n\}$ be an orthonormal basis for $\mathbb{R}^n$, where $\{\mathbf{e}_1, \ldots, \mathbf{e}_k\}$ is a basis for S and $\{\mathbf{e}_{k+1}, \ldots, \mathbf{e}_n\}$ is a basis for $S^\perp$. There exists an integral basis $\{\mathbf{v}_1, \ldots, \mathbf{v}_n\}$ for L such that $\{\mathbf{v}_1, \ldots, \mathbf{v}_k\}$ is an integral basis for E_k. Let $\{\mathbf{w}_1, \ldots, \mathbf{w}_n\}$ be the dual basis, where $\mathbf{v}_i \cdot \mathbf{w}_j = \delta_{ij}$. Then $\{\mathbf{w}_1, \ldots, \mathbf{w}_n\}$ is an integral basis for L_n^* and $\{\mathbf{w}_{k+1}, \ldots, \mathbf{w}_n\}$ is an integral basis for F_{n-k}. Then there's a matrix $\mathbf{M} = \begin{pmatrix} \mathbf{A} & \mathbf{0} \\ \mathbf{B} & \mathbf{C} \end{pmatrix}$ so $\begin{pmatrix} \mathbf{v}_1 \\ \vdots \\ \mathbf{v}_n \end{pmatrix} = \mathbf{M} \begin{pmatrix} \mathbf{e}_1 \\ \vdots \\ \mathbf{e}_n \end{pmatrix}$ and $\begin{pmatrix} \mathbf{w}_1 \\ \vdots \\ \mathbf{w}_n \end{pmatrix} = \left(\mathbf{M}^{-1}\right)^\mathsf{T} \begin{pmatrix} \mathbf{e}_1 \\ \vdots \\ \mathbf{e}_n \end{pmatrix}$. Now $\det L_n = (\det \mathbf{A})^2 (\det \mathbf{C})^2$, $\det E = (\det \mathbf{A})^2$, and $\det F = (\det \mathbf{C})^{-2}$. □

Corollary. *(Mordell's inequality)*

$$\boxed{\delta_{n-1} \geq \frac{1}{2} \delta_n^{\frac{n-2}{n}}}.$$

Proof. Let L_n be an optimal lattice in n dimensions, so $\delta(L_n) = \delta_n$, and choose the scaling such that $\det L_n = 1$. Then $\det L_n^* = 1$, which implies that $\delta(L_n^*) \leq \delta(L_n)$ and $\mu(L_n^*) \leq \mu(L_n)$. Then by the previous theorem, $\mu(L_n^*) =$ the determinant of the densest 1-dimensional section of $L_n^* =$ the determinant of the densest $(n-1)$-dimensional section of L_n. Now using the above claim, $\delta_{n-1} \geq \left(\frac{\mu(L_{n-1})}{4}\right)^{\frac{n-1}{2}} \frac{1}{\sqrt{\det L_{n-1}}}$ and $\delta_n = \left(\frac{\mu(L_n)}{4}\right)^{\frac{n}{2}}$, so

$$\begin{aligned}
\delta_{n-1} &\geq \left(\frac{\mu(L_{n-1})}{4}\right)^{\frac{n-1}{2}} \frac{1}{\sqrt{\det L_{n-1}}} \\
&\geq \left(\frac{\mu(L_n)}{4}\right)^{\frac{n-1}{2}} \frac{1}{\sqrt{\mu(L_n)}} \\
&= \frac{1}{2}\left(\frac{\mu(L_n)}{4}\right)^{\frac{n-1}{2}} \left(\frac{4}{\mu(L_n)}\right)^{-\frac{1}{2}} \\
&= \frac{1}{2}\left(\frac{\mu(L_n)}{4}\right)^{\frac{n-2}{2}} \\
&= \frac{1}{2}\left(\left(\frac{\mu(L_n)}{4}\right)^{\frac{n}{2}}\right)^{\frac{n-2}{n}} \\
&= \frac{1}{2}\delta_n^{\frac{n-2}{n}}.
\end{aligned}$$

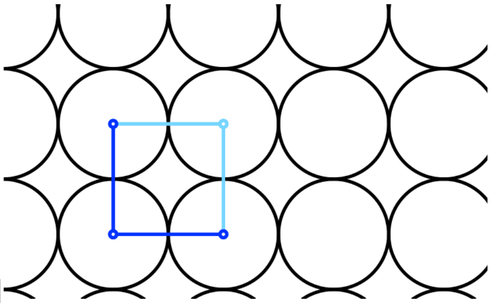
□

Example. Recall that the center density of D_3 is $\delta(D_3) = \frac{1}{4\sqrt{2}}$. Since D_3 is the optimal packing in $\mathbb{R}^3$, we have $\delta_3 = \frac{1}{4\sqrt{2}}$. Then by Mordell's inequality, $\delta_4 \leq (2\delta_3)^{\frac{4}{4-2}} = \frac{1}{8}$. Note that $\delta(D_4) = \frac{1}{8}$, so D_4 is optimal in $\mathbb{R}^4$.

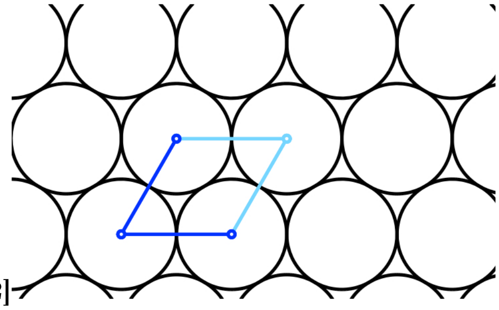
2 Figures

All of these figures were created by the author of these notes.

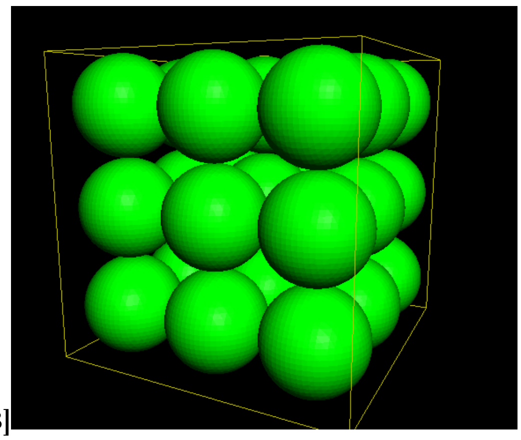
[Figure 1]



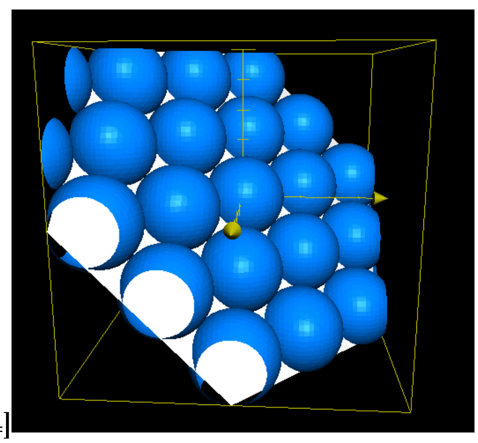
[Figure 2]



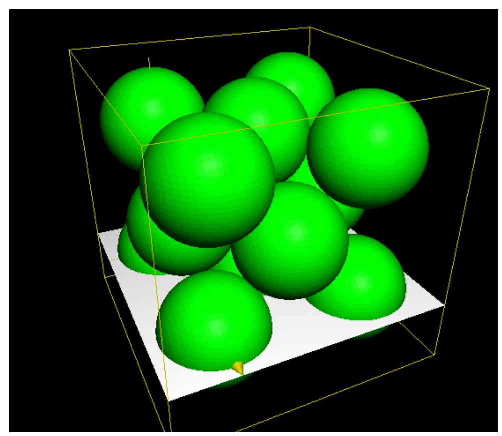
[Figure 3]



[Figure 4]



[Figure 5]



3 References

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