Surreal Numbers and Games

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August 13, 2009

Abstract

Surreal numbers are an alternate number system that constructs the real numbers using ony cuts, and in addition builds some more esoteric numbers. Games are a more general concept that contain the surreal numbers. Surreal numbers have been around since at least the early 1900s, but their modern recognition is largely due to the work of John Horton Conway, spawning from his analysis of endgames for the strategy game Go. In 1974, Donald Knuth's *Surreal Numbers* was published, and was the origin of the term "surreal number". Although surreal numbers can be treated with full blown rigour, this approach is a bit confusing as an introduction. We will therefore opt to approach surreal numbers from their historical beginning in combinatorial games, with some sacrifice of rigour.

1 Games

The word *game* may bring to mind many things, such as tic-tac-toe or monopoly or chess, but we are interested in a very specific class of these. All of the games discussed today will satisfy these stipulations:

1. Two players: We shall call them Left and Right. They alternate making moves.

2. Complete information: If Left knows something about the game in question, Right can deduce it as well, and vice versa. (Chess, for example, is a complete information game. Card games tend not to be, since generally your hand is hidden from the other players.)

3. No chance: Moves have no randomness to them. (This rule eliminates many games, including any game with dice or card shuffling)

4. Winning/Losing Condition: If a player cannot make a move on their turn, they lose.

5. Finite and Determined: The game will come to an end after some finite amount of moves, and the game will result in one player winning and one losing. (Chess has the potential for draws, and so is now off the

table. Many games have cycles of positions that allow a game to continue indefinitely, and this rule eliminates them as well.)

A game generally consists of these two players in a specific "position", where each one has options to "move" to a new "position". We are saying here that a game is characterized by its moveset, the moves that each player can make, each of which will take the players to another, different game:

Definition 1. A *game* G is an ordered pair of sets of games. We often write $G = \{G^L | G^R\}$ where G^L denotes the set of games that Left can move to from G, and similarly for G^R and Right.

This definition seems a bit odd at first: in order to create a game we need some games created, and we don't have those. But we still do know about one set of games, the empty set, and that is enough to make our first game. We shall call it 0.

$$0 = \{ \, | \, \}$$

I shall kindly offer for you to take the first move in this game. Regardless of whether you are playing as Left or as Right, your set of available moves is the empty set and so you cannot move, and I win. Clearly, 0 is a game in which the second player always wins.
Games can be viewed as an inductive process. Knuth's approach was to view 0 as being created on the first "day". On the second day, all the games involving the empty set and 0 that hadn't been previously created were then created. **We will only consider games that are produced by this process,** possibly after an infinite number of days. This gives us a very useful property: the members of a game's pair of sets are simpler, in that they were necessarily created on an earlier day.

2 Blue-Red Hackenbush

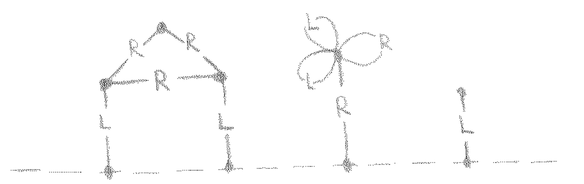

Figure 1: A simple Hackenbush game

Our first example of a game is called Blue-Red Hackenbush, or just Hackenbush for short. This game is composed of red and blue edges that intersect

Figure 2: The zero game in Hackenbush

at their endpoints, all of which are connected to the "ground" either directly or through other edges.

It can easily be checked that Hackenbush satisfies all our conditions for being a game. The zero game can be represented as a Hackenbush game: since in the zero game neither player can make any move, it follows that there can be no blue edges, since otherwise left would have a move if they started, and similarly no red edges. So the zero game is just the ground with nothing attached.

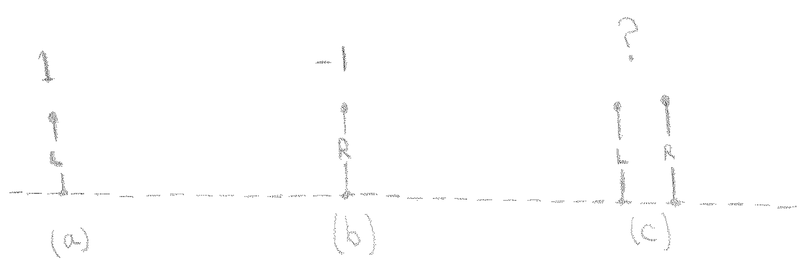

Figure 3: Some simple games

Figure 3 shows a couple of simple Hackenbush games. In (a), Right still has no valid moves, but Left has a single move. Right will lose immediately if they start, whereas if Left starts, he will delete the one blue edge and reduce to the zero game. This can be written as the game $\{0|\}$. In Figure 1, we see this game off to the side. Here it will give Left "one free move" to make during the course of the game. We might as well call it 1. Similarly, we could call (b) -1, since it gives one free move to Right.

$$1 = \{0|\}, -1 = \{|0\}$$

What about (c)? It's the game $\{-1|1\}$. No matter who starts, the second player clearly wins.

3 Equivalence Classes

It's clear that the game of Figure 3c acts very similarly to the game 0. In both games, neither player is eager to start play by going first, for then they will lose. If you were playing a game and saw a copy of 3c, you could simplify by replacing the copy with nothing, since neither player will have any incentive to move in it until they are forced to, and so the person who would lose in the replacement game would move first in 3c and lose anyway. This is true of any game where the second player has a winning strategy. We shall therefore consider games up to equivalence class. Furthermore, our definitions of 1 and -1 give us some motivation for the following definitions of positivity and negativity:

Definition 2. For a game G we say the following:

$G = 0$ iff the second player has a winning strategy.

$G > 0$ iff Left has a winning strategy (regardless of who starts).

$G < 0$ iff Right has a winning strategy.

$G \parallel 0$ ("G is **confused** with 0") iff the first player has a winning strategy.

For surreal numbers, only the first three definitions would be needed, since we have a desire for some kind of trichotomy law, but games are more general and the fourth class is needed. For example, the game $* = \{0|0\}$ is clearly a first player win. We will come back to this game soon.

This consideration of equivalence classes has some significant advantages. For example, if you have a question about how the game 1 acts in a certain situation, you need only consider how the Hackenbush version of it behaves in your situation. Similarly if two games are equal then they are fully interchangeable with each other in a particular situation, because all we really care about is winning.

4 Sums of Games

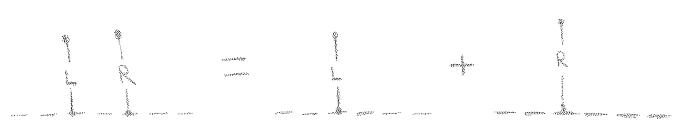

Figure 4: The game $\{-1|1\}$ is the sum of the games 1 and -1

We have already established that the game $\{-1|1\} = 0 = \{\,|\,\}$. If you were to take the games of 1 and -1 and attach their grounds to each other, you would get $\{-1|1\}$. In some way, it is the *sum* of 1 and -1. We often encounter this situation in games we play: when a move in one component of the game has no effect on another component. When we play a sum of games, we pick one of the components and play in only that one.

Definition 3. For $G = \{G^L|G^R\}$ and $H = \{H^L|H^R\}$, we define their *sum* $G + H$ as follows:

$$G + H = \{G^L + H \cup G + H^L|G^R + H \cup G + H^R\}$$

Where $G^X + H$ means the set of sums of one member of G^X with H. We will also write $G + H = \{G^L + H, G + H^L|G^R + H, G + H^R\}$.
We define the *negative* of G as

$$-G = \{-G^R| - G^L\}$$

where $-G^X$ denotes the set of negatives of games in G^X

It should come as no surprise that $1 + -1 = 0$, since this definition merely captures what we already concluded. But it is not immediately obvious that $G + -G = 0$ for any game G, since

$$G + -G = \{G^L + -G, G + -G^R|G^R + -G, G + -G^L\}$$

Theorem 1. *For any game G, $G - G = G + -G = 0$.*

Proof. This is vacuously true for 0, since exchanging sets and negating all members (none) does nothing, and it can easily be checked that $0 + 0 = 0$. Assume it to be true for all games in G^L and G^R (which come before G in the creation process, as was mentioned at the end of section 1). If the first player makes a move in G, the second player can make a corresponding move in -G, and vice versa. For example, Left starts by moving in $-G$ to $-g^R \in -G^R$. Then Right can move in G to $g^R \in G^R$, bringing the game to $-g^R + g^R$, which is zero by the inductive hypothesis. Since 0 was the first game in the creation process, it is therefore true for all games. □

Theorem 2. *For any game G, $G+0=G$.*

Proof. Since we cannot move in 0, we cannot attain the option $G+0^L$ or $G+0^R$. Therefore,

$$G + 0 = \{G^L + 0|G^R + 0\}$$

and this completes the proof by induction. □

Definition 4. For two games G and H, we say the following:

$$\begin{aligned} G = H &\iff G - H = 0 \\ G > H &\iff G - H > 0 \\ G < H &\iff G - H < 0 \\ G \parallel H &\iff G - H \parallel 0 \end{aligned}$$

Exercise. Prove that if $G > 0$, $H > 0$, then $G + H > 0$. Similar results hold for all the cases synonymous with ordering amongst the reals. (Hint: Use game analysis)

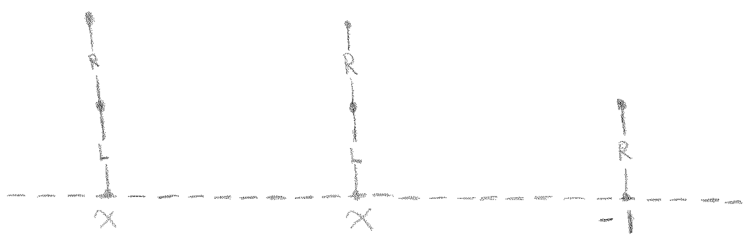

Figure 5: Could these new games be $\frac{1}{2}$?

We will end this section with a look at the newest game in Figure 5, $x = \{0|1\}$. This game is different from all we have seen before. It seems to favor Left, but Right might still have a chance to chop the top node, so it doesn't seem like it would provide a full 1 move advantage. Perhaps half a move?

Exercise. Prove, by analyzing Figure 5, that our new game x satisfies $x+x-1 = 0$, and so $x = \frac{1}{2}$.

Exercise. Prove by induction that a straight line (no branches) of one blue edge connected to the ground followed by n red edges is equal to $\frac{1}{2^n}$.

Exercise. By mimicking the setup of Figure 5, prove that $\{\frac{1}{4}|1\} = \frac{1}{2}$. Conclude that the value of a surreal number (for this is indeed a number as well as a game) of the form $\{a|b\}$ is not necessarily the average of a and b.

5 Large, Small, and Confused Games

The games we have been considering so far have all been rather well behaved. This is a consequence of Red-Blue Hackenbush, which happens to especially well tailored to surreal numbers, but doesn't lead to some of the more esoteric games. Still, we can get some interesting behaviour if we allow games with an infinite number of edges, such as in Figure 6.

The Hackenbush game in 6a is $\omega = \{0, 1, 2, 3, \ldots |\}$. It is clearly larger than any positive finite number n, which is $\{0, 1, 2, \ldots, n-1|\}$ or some equivalent form. In 6b, we have added a single blue edge to the top of (a). Amazingly, this is enough to make it *bigger* than (a)! Similarly, (c) is less than (a):

$$\omega - 1 < \omega < \omega + 1$$

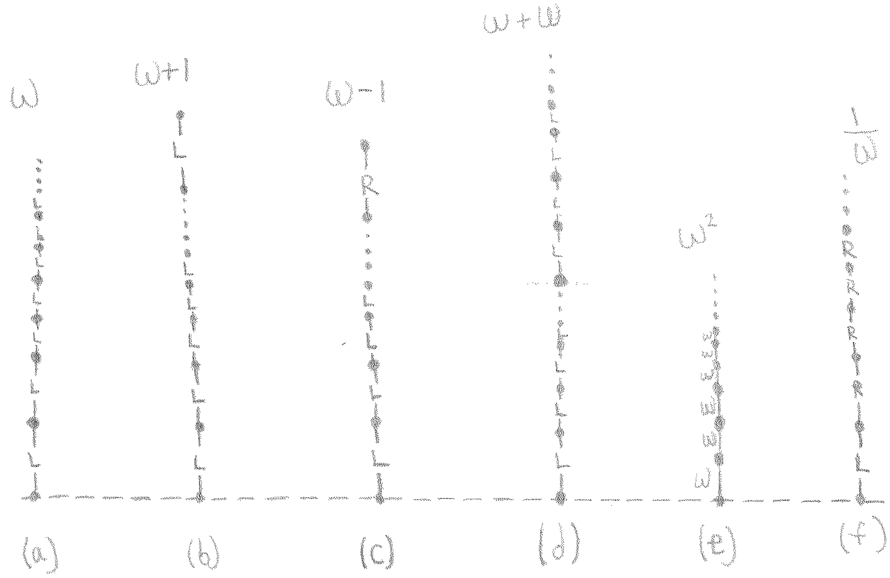

Figure 6: Some infinite games

Stacking two of ω on top of each other gives us (d), which we might as well call $2.\omega = \omega + \omega$. Stacking a countably infinite number of ω's on top of each other gives us (e), which we will call $\omega \cdot \omega = \omega^2$ (there is in fact a multiplication for surreal numbers and games, but we will avoid going into it today). Finally, we state without proof that naming (f) $\frac{1}{\omega}$ is justified by multiplication.

The last one, $\frac{1}{\omega}$ is particularly interesting, since it is the infinite extension of the Hackenbush games corresponding to $\frac{1}{2^n}$. In fact, you can show that $\frac{1}{\omega} < \frac{1}{2^n}$ for every natural number n. This means that $\frac{1}{\omega}$ is smaller than every positive number, and yet it is clearly positive itself. We have the first of many infinitesimal numbers.

5.1 Domineering

We now turn our attention to the game Domineering, in which Left and Right are laying 2 by 1 dominoes on a n by n square grid. Left lays hers vertically, and Right horizontally. As the game progresses, the dominoes will start to wall off smaller sections of the board, and the game will turn into a sum of several smaller games.

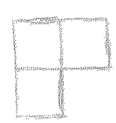

Figure 7: The game $* = \{0|0\}$.

In this Domineering position, whoever plays first in it will stop any further play. Taken alone as a game it will result in a first player win. We call it $*$, and we have just observed that $* \parallel 0$.

Exercise. Prove that * is less than every positive number and greater than every negative number by playing $* - \frac{1}{2^n}$ and $* + \frac{1}{2^n}$.

Exercise. Prove that $* + * = 0$.

$*$ is in fact a *nimber*, called so because of its relation to the game of Nim. Nimbers have their own addition and multiplication rules that are very different from those of ordinary numbers, and as games they are all infinitesimally small.

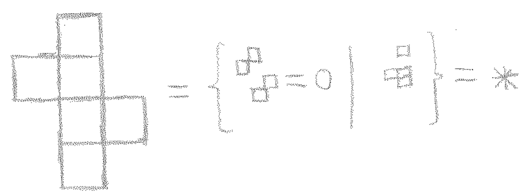

Figure 8: The game $\uparrow = \{0|*\}$

In the game of Figure 8, Left has only one intelligent move, and that is to 0. Right's only possible move is to the position $*$. Clearly this game is positive. We shall call it $\uparrow$ ("up"), and its negative $\downarrow = \{*|0\}$ ("down").

Exercise. Prove that $\uparrow$ is less than any positive number. $\downarrow$ is similarly greater than any negative number. We have another example of a positive infinitesimal.

Exercise. How does $\frac{1}{\omega}$ compare to $\uparrow$? How does it compare to $2.\uparrow = \uparrow + \uparrow$?

Exercise. Analyze $\uparrow +*$ and show that $\uparrow +* \parallel 0$. Does the same hold for $\downarrow +*$? What does this say about $*$?

We close with a discussion of the game in Figure 9, which we denote ± 1 (in general, $\pm G = \{G| - G\}$). Whoever plays first, their move will secure an additional move for them and none for the other player, so $\pm 1 \parallel 0$. What about $1 \pm 1 = 1 + (\pm 1)$? If Left moves first, he can move to 2 and secure a win. If Right moves first, they will move to $1 + -1 = 0$ and win! So $1 \pm 1 \parallel 0 \implies \pm 1 \parallel -1$ and similarly $\pm 1 \parallel 1$. In fact, it can easily be shown that for any number x in $[-1, 1]$, $\pm 1 \parallel x$.

Exercise. Prove that for any number x in $(1, \infty)$, $\pm 1 < x$. Similarly, $x < \pm 1$ for any x in $(-\infty, -1)$. This shows that ± 1 has a sort of "confusion interval" of numbers that it is confused with, as in Figure 9.

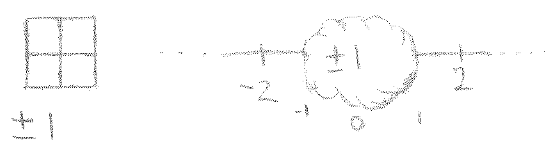

Figure 9: The game $\pm 1 = \{1|-1\}$, and its confusion interval.

Exercise. From a previous exercise, $\uparrow + * \parallel 0 \implies \uparrow \parallel *$, and $\downarrow \parallel *$. What does this imply about the confusion interval of $*$? Is it true that $* \parallel \uparrow + \uparrow = 2.\uparrow = \Uparrow$?

References

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[3] Knuth, *Surreal Numbers*, Addison-Wesley Publishing Company Inc., United States (1974)