What is an ultrafilter?
John Johnson, July 9, 2013

An ultrafilter, on some fixed nonempty set X, can be roughly characterized as a combinatorial measure on X. The study of ultrafilters, then, is essentially the study of combinatorial measure theory. This type of measure theory has several "good" and "bad" points. (Depending on your mathematical point-of-view and interests, the "good" can be bad or the "bad can be good.)

The "good" points:

1) every subset of X is measurable;
2) when X is infinite, there exists a nontrivial measure — where nontrivial means that the measure has no positive point-mass;
3) strongly connected with the above point, ultrafilters can characterize nonempty finite sets; and
4) when X is equipped with an algebraic structure, there exists special ultrafilters which allow us to state and use an analogue of the Poincare recurrence theorem from classical measure theory.

The "bad" points:

1) every subset of X either has measure 0 or measure 1;
2) the existence of a measure without point-mass depends on your underlying axiomatization of set theory;
3) similar to many characterizations of finite sets, the characterization involving ultrafilters depends on a version of the axiom of choice.
4) if we ask for a slightly stronger version of an analogue to Poincaré recurrence theorem, then the existence of these types of ultrafilters are beyond the usual widely accepted axiomatizations of set theory.

In this presentation, I will give several definitions of an ultrafilter, present the analogy between Poincaré recurrence theorem and Hindman's theorem (the latter can be viewed as a combinatorial version of Poincaré recurrence), and indicate how various types of ultrafilters are used in mathematics.