Van Schooten's Locus Problem

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Abstract

Van Schooten's Locus Problem asks: "Two vertices of a rigid triangle in a plane slide along the arms of an angle of the plane. What locus does the third vertex describe?" We will solve this problem and, in doing so, find two constructions for an ellipse.

If time permits, we will address some additional topics. Here is the tentative list:

1) Steiner's Division of Space by Planes problem which asks: "What is the maximum number of parts into which a space can be divided by n planes?"

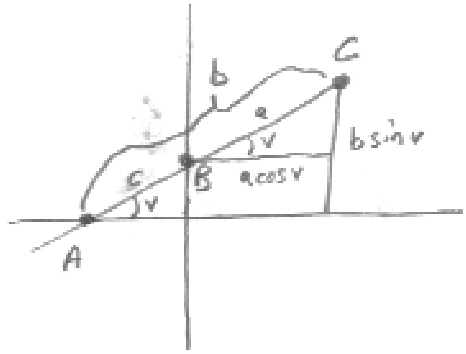
2) Euler's Tetrahedron Problem which asks us to express the volume of a tetrahedron in terms of its six edges.

Van Schooten's Locus Problem

Background: This problem was solved and published by Franciscus van Schooten (1615-1660), a Dutch mathematician who used analytic geometry. The problem was put forth in van Schooten's *Exercitationes mathematicae* (1657).

We begin with a preliminary result: given three rigid points on a line, let two of these points slide along the arms of a right angle. What is the locus of the third point? This case was known well before van Schooten—Proclus (Byzantine, 410-485) had taught the solution.

Let the right angle form the x- and y-axes of our coordinate system. Let the line have the points A, B, and C marked, with A sliding on the x-axis and B on the y-axis. Let BC=a, CA=b, and AB=c. Thus, c = a +/- b, depending on where C lies. We write C's location in coordinates as (x,y). Let v be the angle of the line with respect to the x-axis. Then, as x is the projection of a onto the x-axis, x= a cos v. Similarly, y is the projection of b onto the y-axis, so y = b sin v.
Thus, (x/a)^2 + (y/b)^2 = 1, so the locus of C is an ellipse with half axes a and b.



This property gives us two constructions for an ellipse:

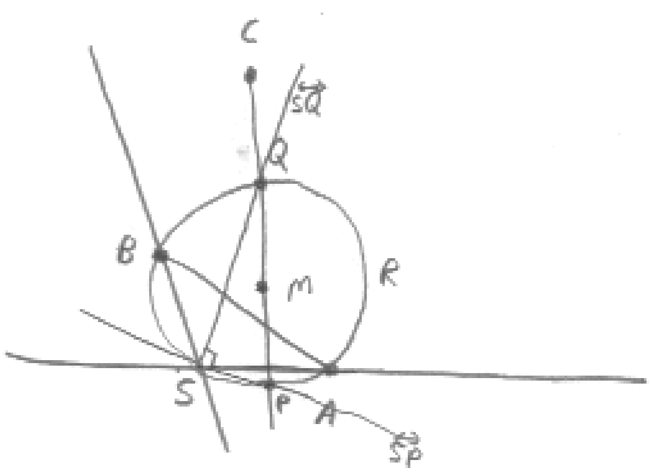
1) Paper Strip Construction

On a paper strip, we mark off three points, A, B, and C. We move the strip such that A is always on the x-axis and B is always on the y-axis. By marking C's location, we obtain an ellipse.

2) The Trammel

The trammel is a device with two grooves at right angles to each other and a point confined to move within each groove. These points are connected by a rigid object which can hold a pencil at a third point. Because the two inner points are confined to move on the trammel's axes, the pencil will trace out an ellipse.

We now solve the general van Schooten Locus Problem. Let the triangle be ABC where C is the point not attached to a leg of our angle. Let the apex of this angle be S. Draw the circle R that contains the points A, B, and S. Let its midpoint be denoted M, and join M and C along a line that crosses through R twice. Denote the intersection points as P and Q. Now, when moving our triangle, consider this circle to be attached. As angle ASB is the peripheral angle opposite AB, R passes continuously through S. We note that the arcs AP and AQ will continuously change their position, but not their magnitude. Thus, the peripheral angles ASP and ASQ are invariant, so the directions given by SP and SQ are also invariant (as A is confined to one leg of our initial angle). We note that PQ is a diameter of R, so these directions (SP and SQ) are perpendicular. Thus, we can consider the motion of C as being a part of the line PQC sliding along the arms of the directions given by SP and SQ. By our preliminary result, this is an ellipse.



Steiner's Division of Space by Planes

Background: Jakob Steiner was a Swiss mathematician who specialized in geometry. This problem was treated in Steiner's "Several laws governing the division of planes and space".

We again begin with a preliminary result: what is the maximum number of parts into which a plane can be divided by n straight lines?

To maximize the number of parts, we will take no two lines to be parallel and no more than two lines to pass through a single point. It is clear that if either of these were to be the case, then a slight adjustment of a line would introduce more parts.

Let the plane be divided by n lines into $\bar{n}$ parts. We draw an additional line. This line is divided by the first n lines into n points, and so it crosses n+1 of the possible $\bar{n}$ parts of the plane. Each crossing divides these parts into two more parts, so we increase the number of parts by n+1. Thus, we have $\overline{n+1} = \bar{n} + (n+1)$. For n = 0, 1, ... this produces the set of equations

$$\bar{1} = 1 + 1$$
$$\bar{2} = \bar{1} + 2$$
$$\ldots$$
$$\bar{n} = \overline{n-1} + n$$

Adding these equations gives us

$$\bar{n} = 1 + (1 + 2 + \ldots + n) = 1 + [n\,(n+1)]/2$$

We now do the same for the three-dimensional problem. To maximize parts, we let no more than three planes intersect at one point and have the lines of intersection of no more than two planes be parallel. Given this, we denote the number of pieces our n planes divide R^3 into as $\tilde{n}$.

As before, we add another plane to our set of planes. This plane is cut by our first n planes into n lines, no more than two passing through a single point and no two or more being parallel. This new plane is thus divided by the n lines into $\bar{n}$ sections.

Thus, $\widetilde{n+1} = \tilde{n} + \bar{n}$. For n = 1, 2, ... this gives the set of equations

$$\tilde{1} = 1 + 1$$
$$\tilde{2} = \tilde{1} + \bar{1}$$
$$\tilde{3} = \tilde{2} + \bar{2}$$
$$\ldots$$
$$\tilde{n} = \widetilde{n-1} + \overline{n-1}.$$

Adding gives us

$$\tilde{n} = 2 + \bar{1} + \bar{2} + \ldots + \overline{n-1}$$

So, by our preliminary result,

$$\tilde{n} = n + 1 + [1/2][(1)(2) + (2)(3) + \ldots + (n - 1)(n)]$$

Simplification gives

$$\tilde{n} = n + 1 + [1/2]\ \{[1^2 + 2^2 + \ldots + (n - 1)^2] + [1 + 2 + \ldots + (n - 1)]\}$$

so

$$\tilde{n} = [1/6]\ [n^3 + 5n + 6].$$

References

[1] Dorrie, Heinrich translated by David Antin, *100 Great Problems of Elementary Mathematics Their History and Solution*, Dover Publications, Inc., New York, New York (1965)

[2] Wikipedia (http://en.wikipedia.org/wiki/Frans_van_Schooten, http://en.wikipedia.org/wiki/Jakob_Steiner)