What is... Crofton's Formula?

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Using lines to approximate curves is an age-old technique in mathematics. Archimedes used it to estimate the value of π, and it underlies calculus. In this talk, we'll explore another application: Crofton's formula, which relates the arc length of curves to the measure of infinitely many lines that intersect the curve. This result has some advantages over other common arc-length formulas, and it has far-reaching consequences into Riemannian geometry, probability, and real analysis.

We'll go over the historical context in which Morgan Crofton published the formula (though it was known earlier by Cauchy), the intuition behind the result, the details of the statement, one scenario where it differs from the "usual" arc length formula, a clever numerical approximation that has been used effectively in biology, and some applications to classical geometry problems.

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OSU "What is… ?" Seminar

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Morgan Crofton

- 1826-1915
- Geometric Probability Theory
- Trinity College, Dublin
- James Sylvester
- *On the Theory of Local Probability* (1868)

Image Source: MacTutor

Outline

- Review of Measure
- Intuition of the Result
- Statement of the Theorem
- Proof of the Theorem
- Applications
- Examples

Real Analysis Review: Measure

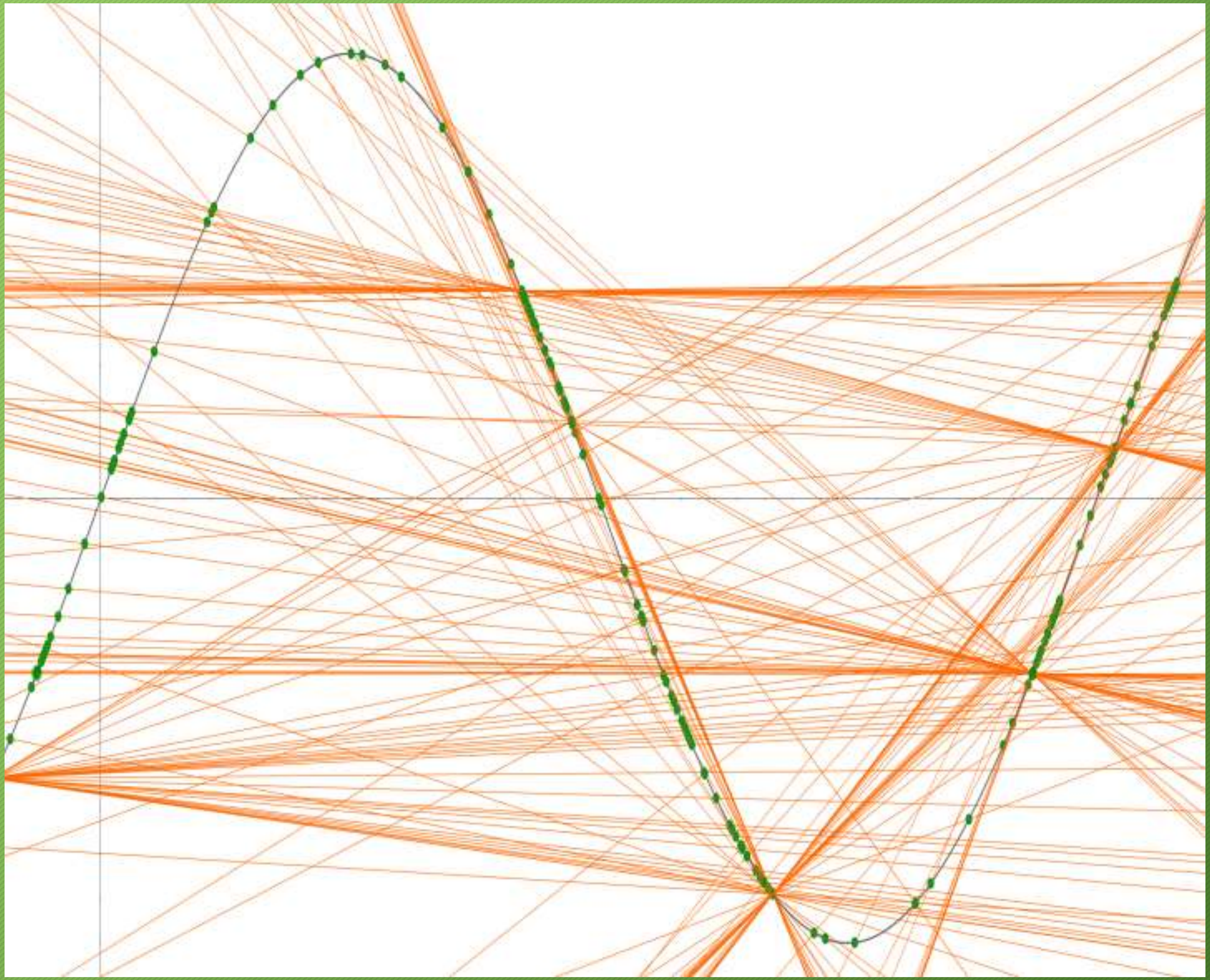
A function that assigns a “size” to given sets

- Nonnegative
- Empty set always has measure (size) zero
- Additive over disjoint sets

We’ll measure infinite collections of lines in the coordinate plane….

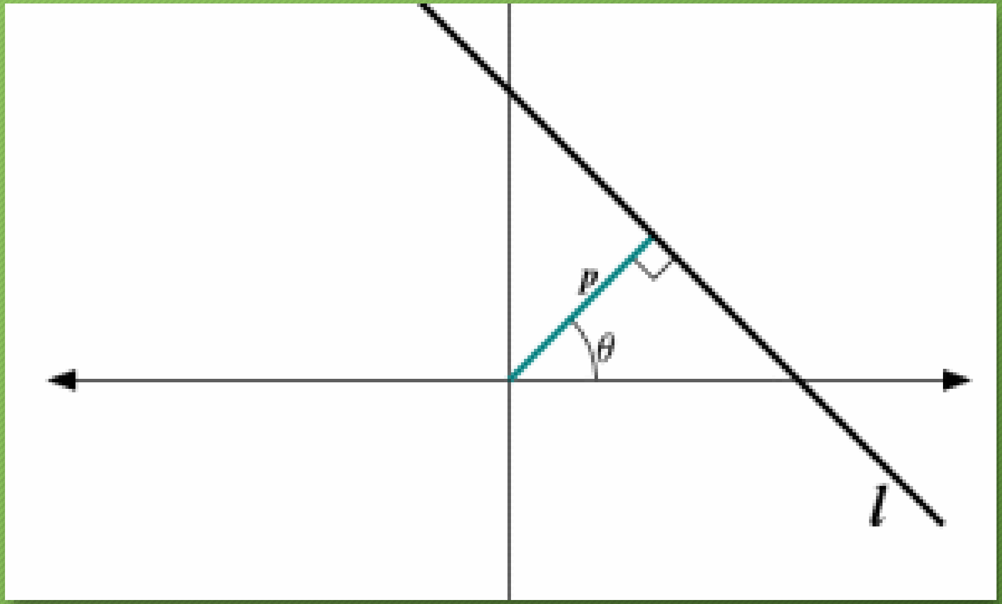
Intuition: "Battleship Meets Pick-up Sticks"

- Length of a curve is a "summation" of its points
- Identify points in polar coordinates and add 'em up
- We don't care *where* the intersections (green points) occur, just how many



How do we keep track of our lines?

- Any line in $\mathbb{R}^2$ can be described by exactly two pieces of information.
- Given a line, we get a certain number of "pings" depending on how many times it intersects our "opponent's" curve.
- Define this function $n(\ell) = n(p, \theta)$



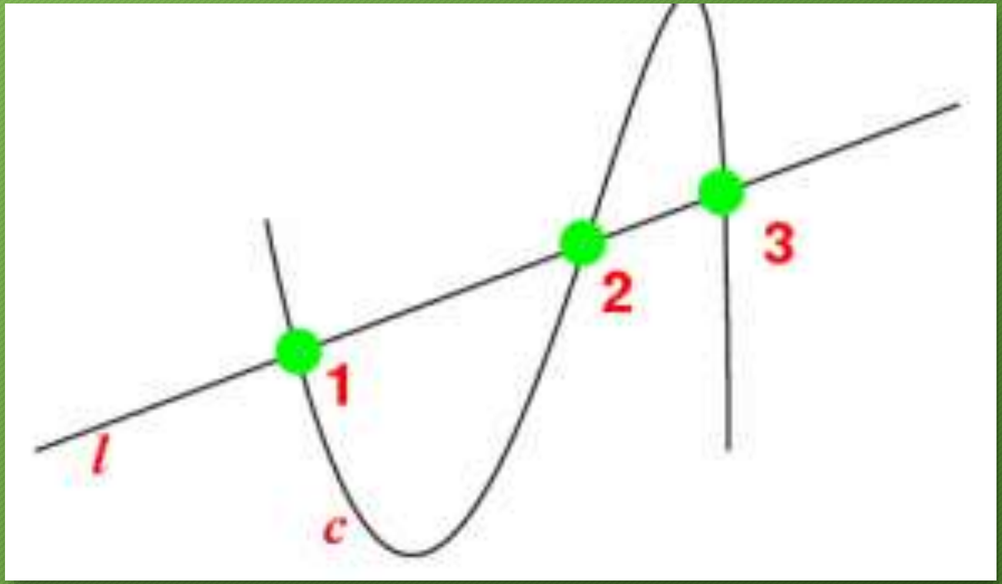


Image source: Adam Weyhaupt (reproduced with permission)

How about "adding 'em up?"

- The previous definition is nudging us towards polar coordinates.
- Define our measure μ as $\mu(S) = \int \int_S dp\, \mathrm{d}\theta$. Note: it is invariant under plane isometries.
- The set S will be the collection of lines that intersect the curve (counting multiple intersections)

Crofton's Formula

Let $\gamma: [0,1] \to \mathbb{R}^2$ be a regular plane curve.

γ is differentiable and $\gamma'(t) \neq 0$ on all of $[0,1]$

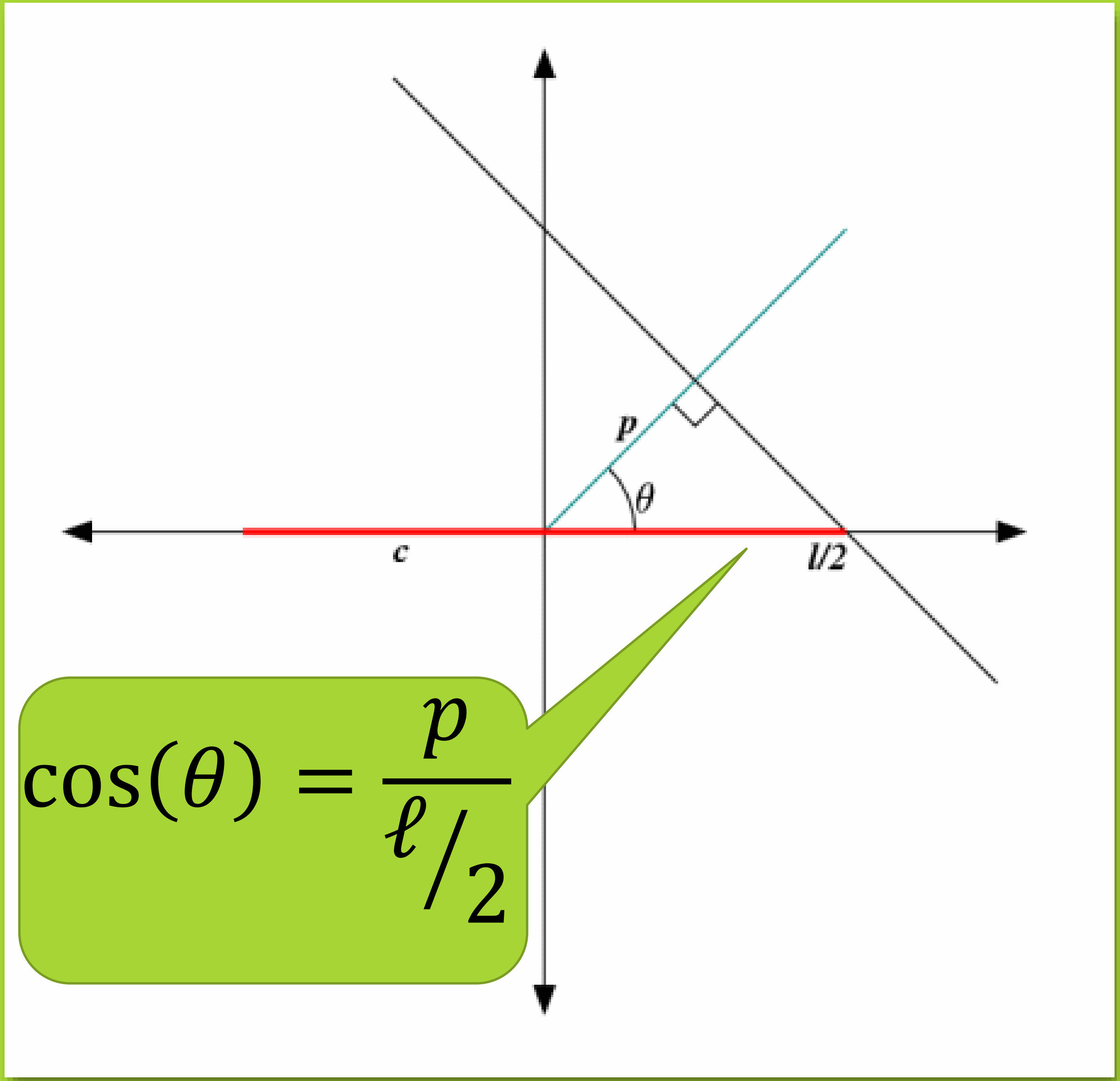
Then

$$\text{Len}(\gamma) = \frac{1}{2} \iint n(p, \theta) \; dp \; d\theta$$

Proof: Step 1

- Crofton holds for line segments (WLOG, center at origin)

$$\frac{1}{2}\int_0^{2\pi}\int_0^{\left(\ell/2\right)|\cos\theta|} dp\,d\theta$$



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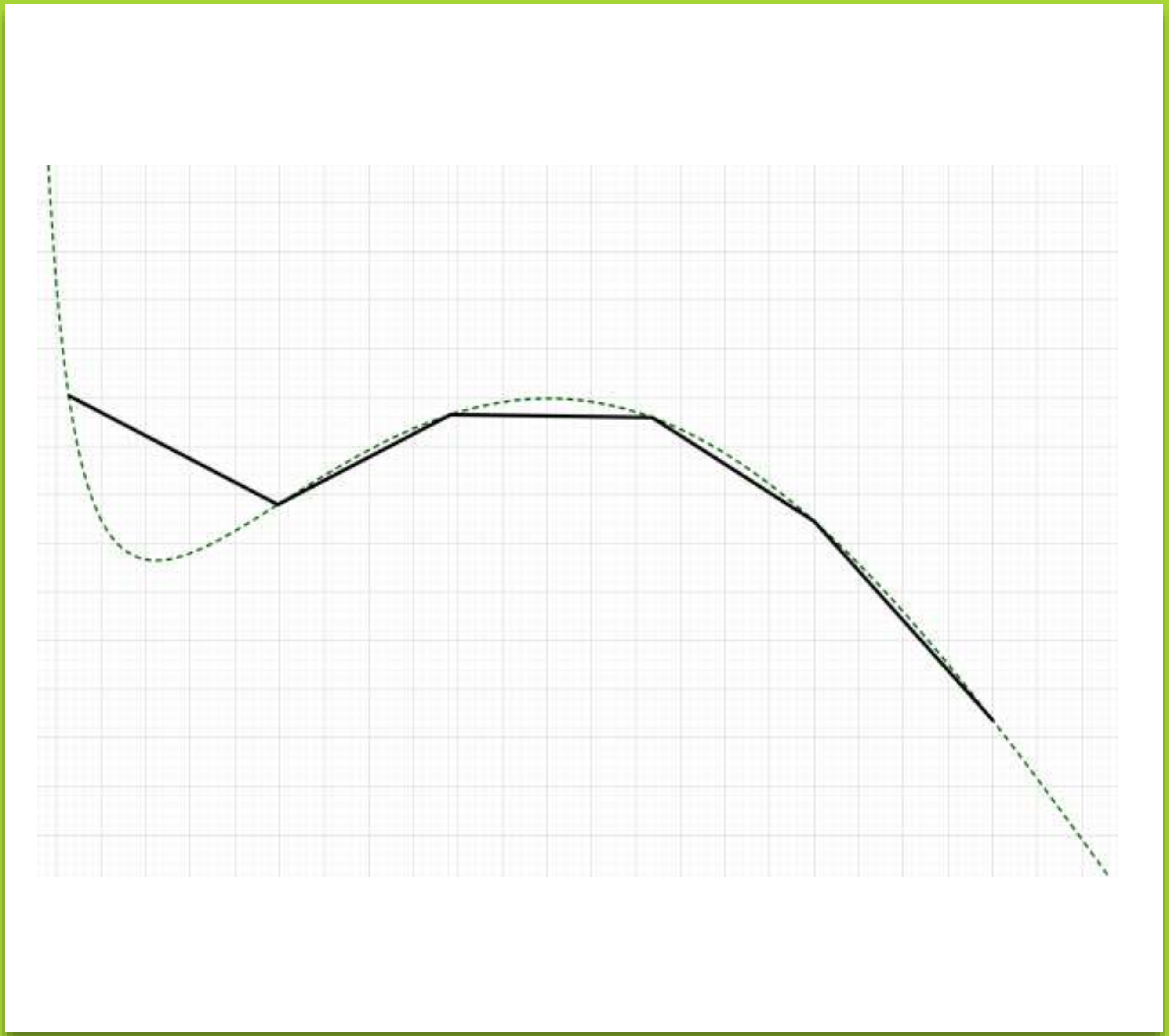
Proof: Step 2

- Integrals are additive
- Regular curves are the limit of piecewise-linear curves



Example: Circle Perimeter

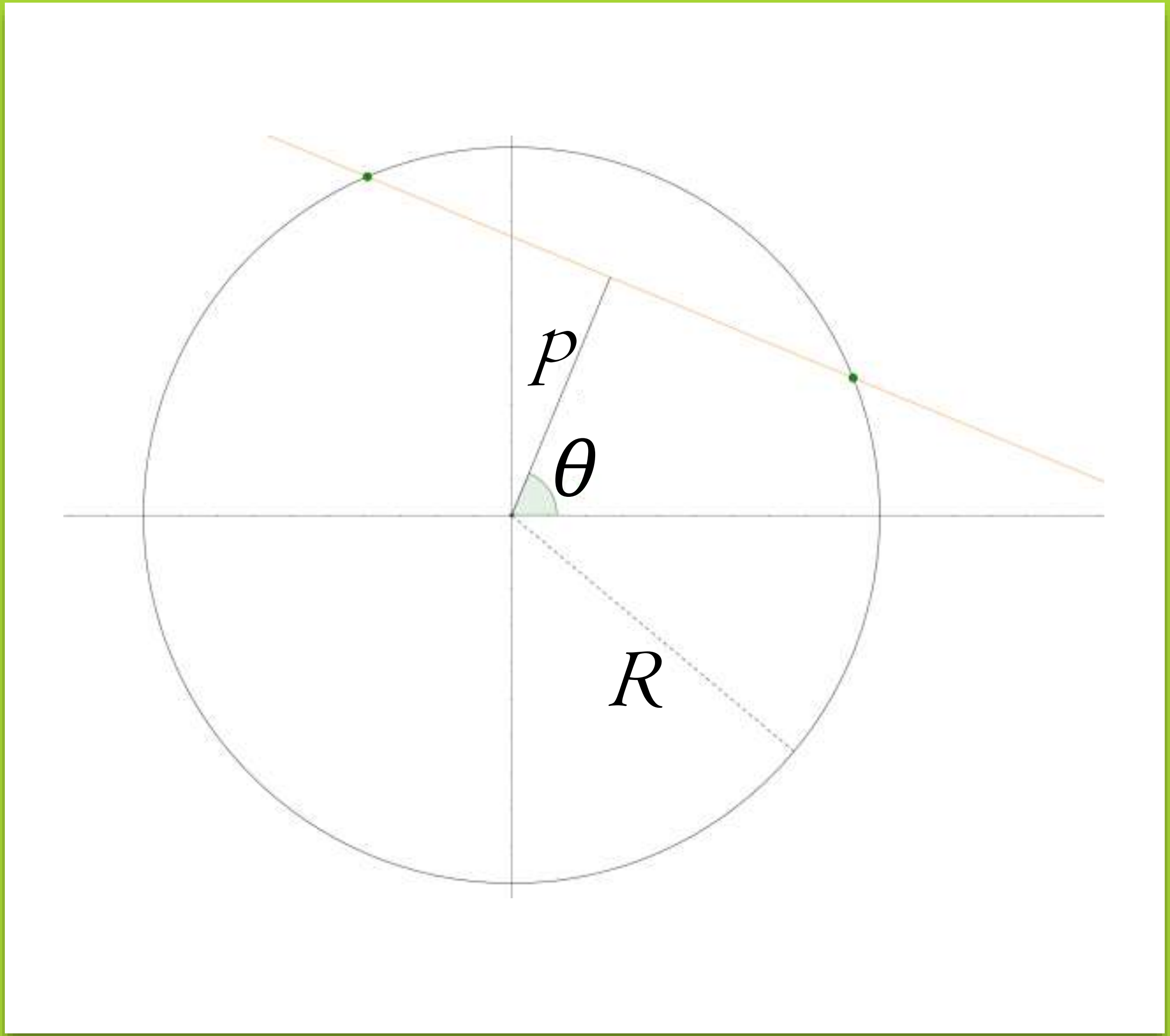
Intersection function:

$$n(p,\theta) = \begin{cases} 0, p > R \\ 2, p \le R \end{cases}$$

Hence, the length is

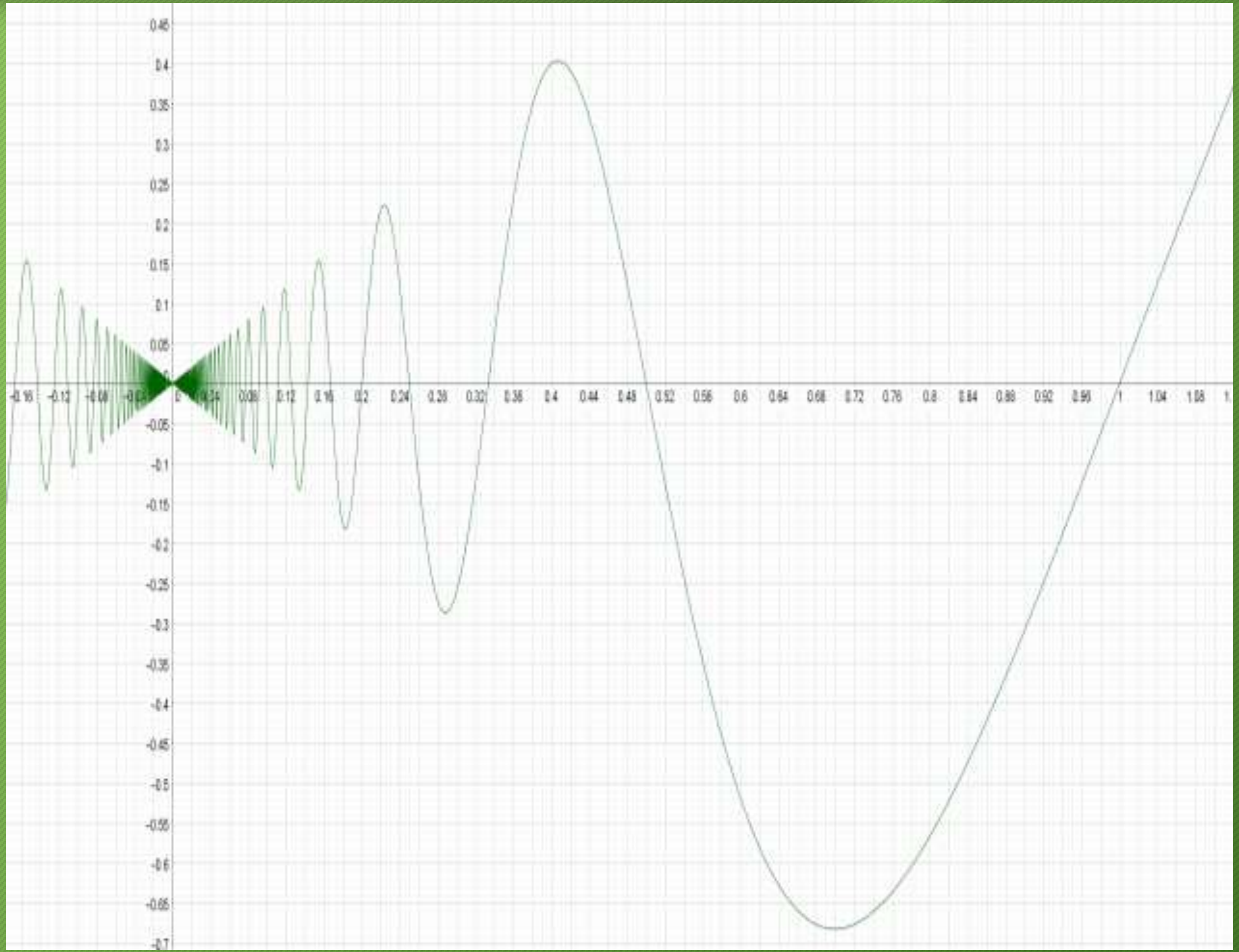
$$\frac{1}{2}\int_0^{2\pi}\int_0^R 2\, dp\, d\theta$$

$$= 2\pi R$$



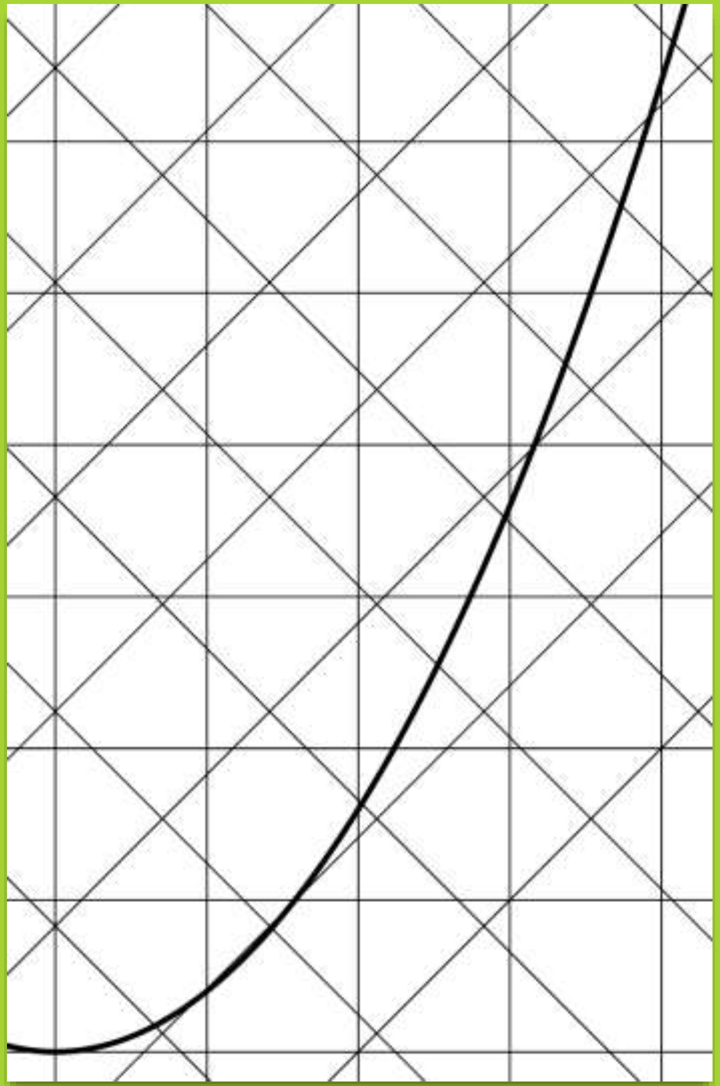
Example: a Non-rectifiable Curve

- $\gamma(t) = t \sin\left(\frac{\pi}{t}\right), t \in [0,1]$
- Arc length infinite, but…
- Crofton formula is finite



Numerical Approximation

- Difficult to compute $n(r, \theta)$?
- Create a "mesh" of lines, $r = 2$
- How many intersections? $n = 20$
- Claim: the curve length is approximately $\frac{1}{2} nr \frac{\pi}{4} \approx 15.708$
- Actual length ≈ 15.760
- Application: bacterial DNA length



Applications

- Isoperimetric Inequality: the area A enclosed by a curve of perimeter L satisfies

$$4\pi A \leq L^2$$

  - Corollary: circles enclose the most area.

- Barbier's Theorem: curves of constant ***width*** $\boldsymbol{w}$ **have perimeter** $\boldsymbol{\pi w}$**.**

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