What is... Game Theory?

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ABSTRACT

Game theory is a branch of mathematics used primarily in economics, political science, and psychology. This talk will define what a game is and discuss a variety of ways in which games can be classified and described. Later, I will introduce a concept known as the Nash equilibrium, which is a state of a game in which no player has incentive to change his strategy after considering the strategies of other players.

WHAT IS GAME THEORY?

A *game*, in the mathematical sense, is a situation in which players make rational decisions according to defined rules in an attempt to receive some sort of payoff. *Game theory* is the branch of mathematics which focuses on the analysis of such games. Game theory can be divided into two main subdisciplines: classical game theory and combinatorial game theory.

Classical game theory studies games in which players move, bet, or strategize simultaneously. As a result, players often find themselves ignorant to certain aspects of the game. Players of these games are more likely to depend on prediction and chance due to this lack of information. Examples include poker or rock, paper, scissors.

Combinatorial game theory, on the other hand, is the study of two-player games in which each player has complete knowledge of all aspects of the game throughout the entirety of gameplay. These games are usually played on a turn-by-turn basis and do not typically involve elements of chance. Examples include chess or checkers. Furthermore, combinatorial games are said to be *impartial* if all players have the same set of possible moves from each position. Otherwise, the game is said to be *partizan*.

CLASSIFICATION OF GAMES

In addition to the two classifications presented above, games can be classified in a variety of ways. One of the most obvious is to classify a game by the number of players. It is common to describe a game as an n-person game, where n is an integer greater than or equal to 1 representing the number of players required to participate in a particular game.

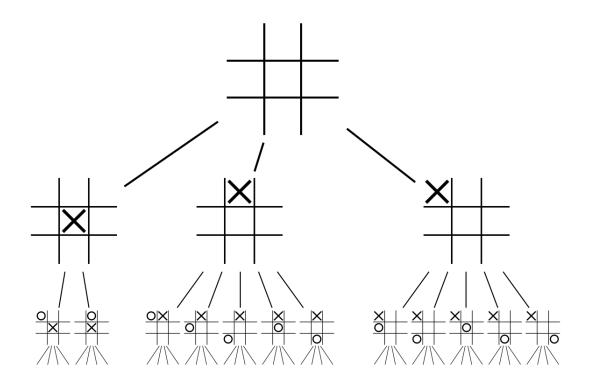
The order in which players move (or lack thereof) is another easy way to classify games. Players all make their move at the same time in a *simultaneous game*. Contrarily, in a *sequential game*, only one player may move at any given time. Some games may not necessarily fall into either of these categories.

Games can also be classified based on the total possible winnings. A *constant-sum game* or *zero-sum game* is one in which the sum of total possible winnings remains constant no matter what actions the players take; that is, the sum of the winnings gained by some players must be equal to the sum of the other players' losses. In poker, for example, players compete for a constant sum of money. The decisions of each player do not affect the available winnings. In *variable-sum games*, however, the total available winnings may change depending on the payers' actions. The prisoner's dilemma is an example of a variable-sum game.

Variable-sum games can be divided even further into the following subgroups: cooperative and non-cooperative games. Players of *cooperative games* are permitted to make binding agreements, such as an enforceable contract, while players of *non-cooperative games* may not create any binding arrangements. For example, imagine there are two individuals, a seller and a buyer, hoping to complete a business transaction. As they attempt to negotiate a price, the individuals are participating in a non-cooperative game. If the buyer signs a contract agreeing to pay a specific price, it then becomes a cooperative game.

REPRESENTATION OF GAMES

There are a variety of ways in which we may describe games. The first we will discuss is known as *extensive form*. In this method, the sequence of choices made in a game are depicted using a game tree. The payoff for each possible sequence is noted at the end of each of the final branches. Consider, for example, the following partial game tree for a game of Tic-Tac-Toe between two players:



Since rotations and reflections are equivalent, Player 1 has three possible moves, as pictured above. Player 2's possible moves change depending on the choice made by Player 1. If we were to further extend the tree, we would see what moves Player 1 could make following Player 2's decision. We could draw a complete tree to see all possible outcomes from all possible move sequences. Note that extensive form can be used to describe simultaneous games as well by using dashed lines to indicate that a player is unaware of which node he is in.

Although games can be validly described in extensive form, they may be more clearly described using *normal form*, also known as *strategic form*. For this reason, normal form is more commonly used to describe simultaneous games (typically with two players). In normal form, a game is represented using a matrix which describes the outcomes for both players for any combination of moves. For example, consider the following matrix depicting the prisoner's dilemma game:

		PRISONER 2	
		Confess	Lie
PRISONER 1	Confess	-8,-8	0,-10
	Lie	-10,0	-1 ,-1

The players and all of their possible moves are placed on adjacent sides of the matrix. The payoffs are placed inside the matrix. In this case, all of the payoffs are negative, since they represent time spent in jail. From the matrix pictured above, we see that if both prisoners confess, they each spend 8 years in jail. If only one prisoner confesses, he walks away with no jail time, while the other player must spend 10 years in jail. If both prisoners lie, they each spend only 1 year in jail.

Finally, cooperative games can be represented in characteristic function form. This method is somewhat different than the other two in that it examines the payoff for the group of players as a whole rather than considering individual decisions and payoffs. Denote the (finite) set of all players of a game by $N = \{1, 2, ..., n\}$. A *coalition* $C \subseteq N$ is any subset of players. Note that we are going to assume *transferable utility*, meaning we can assign one payoff to the whole set of players N (or any coalition C) and allow it to later be arbitrarily distributed among the individual players within the group. Under this assumption, we may define the *characteristic function form* as a tuple $\langle N, v \rangle$, where N is the set of all players (as previously stated) and $v: 2^N \to \mathbb{R}$ is a function which maps every possible coalition $C \subseteq N$ to its payoff v(C). Assume $v(\emptyset) = 0$. The characteristic function of a cooperative game describes the collective payoff a set of players might receive by forming a coalition. It helps us decide which coalitions should form in order to receive the largest (collective) payoff. I will provide an example of this on the next page.

Example. Consider a set of players $N = \{1,2,3\}$. Players 1 and 2 each possess a right-hand glove, while Player 3 possesses a left-hand glove. The gloves are worth nothing on their own, but a pair (consisting of a right- and left-hand glove) is worth \$10. We can construct the characteristic function $\langle N, \nu \rangle$ as follows:

$$v(\emptyset) = 0$$

$$v(\{1\}) = v(\{2\}) = v(\{3\}) = 0$$

$$v(\{1,2\}) = 0$$

$$v(\{1,3\}) = v(\{2,3\}) = 10$$

$$v(N) = 10$$

It is easy to see from the above that Players 1, 2, and 3 all gain nothing if they do not form coalitions. Moreover, Players 1 and 2 receive no benefit by forming a coalition with each other. However, if Players 1 and 3 or 2 and 3 form a coalition with each other, they receive a collective payoff of \$10. The same is true for the entire set of players N. Looking at the characteristic function v allows us to more easily determine if it would be beneficial to form coalitions, and, if so, which coalition(s) would result in the highest collective payoff.

THE NASH EQUILIBRIUM

The Nash equilibrium is a concept which was originally presented by American mathematician John Nash (1928-2015). A non-cooperative game is said to be in *Nash equilibrium* if no player has incentive to change his individual game strategy after considering the strategies of all other players. The prisoner's dilemma is a classic example of the Nash equilibrium. As a reminder, the prisoner's dilemma is a situation in which two prisoners are convicted as accomplices in a crime. The prisoners are placed in solitary confinement, so they have no method of communicating with each other. They are then each presented with the following proposal:

- i. If they both confess, they will each spend 8 years in jail.
- ii. If only one of them confesses, he will be set free while the other will spend 10 years in jail.
- iii. If neither of them confesses, they will each spend 1 year in jail.

This game is in Nash equilibrium when both prisoners confess. Why? Because under these circumstances, neither prisoner benefits by changing his strategy. If Prisoner 1 were to change his strategy and instead keep quiet, then he would receive a longer jail sentence than he would if he confessed, and Prisoner 2 will be able to walk away without punishment. Prisoner 2 should maintain his strategy as well by the same logic. Although the best strategy for the group as a whole would be for both to keep quiet, individually the prisoners are better off confessing since they have no way of knowing the other prisoner's strategy beforehand, and staying quiet while the other confesses would result in 10 years of jail time.

The Nash equilibrium can be applied to a variety of real-life situations. It explains, for example, why we overfish the seas: Although overfishing is clearly bad for the ecosystem as a whole, it would be bad for an individual company to stop fishing because then that company would stop profiting while other companies continue to fish and, hence, continue to make a profit. The Nash equilibrium can also be applied in economics, war, politics, and countless other fields.

RESOURCES

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