What are the Markov and Lagrange Spectra?

James Enouen

July 2018

1 Introduction to Approximation by Rationals

1.1 Dirichlet's Approximation Theorem (c. 1840)

For any real number α, there are infinitely many p, q $\in \mathbf{Z}$ such that:

$$\left|\alpha - \frac{p}{q}\right| < \frac{1}{q^2}$$

meaning that there are infinitely many rational approximations which are good with respect to how finely they divide $\mathbf{R}$ or how large their denominator is.

1.2 Hurwitz's Approximation Theorem (1891)

For any real number α, there are infinitely many p, q $\in \mathbf{Z}$ such that:

$$\left|\alpha - \frac{p}{q}\right| < \frac{1}{\sqrt{5}q^2}$$

This $\sqrt{5}$ is actually the best constant we can put at this location, because for any greater constant on the denominator, the statement will no longer be true for the golden ratio, $\phi = \frac{1+\sqrt{5}}{2}$.

1.3 Diophantine Approximation of Algebraics

Many people started to investigate the diophantine approximation on algebraic irrationals. This began with Liouville who showed that for an irrational algebraic α of degree d, there is a constant $C > 0$ where for all $\frac{p}{q} \in \mathbf{Q}$

$$\frac{C}{q^d} < \left|\alpha - \frac{p}{q}\right|$$

This result states that an algebraic of degree d can not be well approximated to any degree greater than d.

Below is listed a series of results of the same flavor ultimately leading to Roth's Theorem.

Author	Date	Order of Approximation
Liouville	1844	$\leq d$
Thue	1909	$\leq d/2+1$
Seigel	1921	$\leq 2\sqrt{d}$
Dyson	1947	$\leq \sqrt{2d}$
Roth	1955	≤ 2

Roth's Theorem on Diophantine Approximation can also be read: Let $\alpha \in \mathbf{R}$ and $\epsilon > 0$ then if there are infinitely many $\frac{p}{q} \in \mathbf{Q}$ where

$$\left|\alpha - \frac{p}{q}\right| < \frac{1}{q^{2+\epsilon}}$$

then α must be a transcendental number.

I'll also note here that a real number α is called *badly approximable* if there exists a constant c > 0 where $\left|\alpha - \frac{p}{q}\right| > \frac{c}{q^2}$ for all $\frac{p}{q} \in \mathbf{Q} \setminus \{\alpha\}$. This occurs if and only if the continued fraction coefficients are bounded.

2 Defining the Spectra

2.1 The Lagrange Spectrum

Let $\alpha \in \mathbf{R}$. Consider L > 0 such that

$$\left|\alpha - \frac{p}{q}\right| < \frac{1}{Lq^2}$$

holds for infinitely many $\frac{p}{q} \in \mathbf{Q}$. We define L($\alpha$) to be sup(L) over all L satisfying this inequality and call it the Lagrange number of α.
$\mathcal{L} = \{L(\alpha) : \alpha \in \mathbf{R} \setminus \mathbf{Q}\}$ is called the Lagrange spectrum.

Suppose $\alpha = [a_0; a_1, a_2, a_3, ...] = [a_0; a_1, ..., a_n, \alpha_{n+1}]$, then

$$\left|\alpha - \frac{p_n}{q_n}\right| = \frac{1}{q_n(\alpha_{n+1}q_n + q_{n-1})} = \frac{1}{(\alpha_{n+1} + \frac{q_{n-1}}{q_n})q_n^2}$$

Set $\lambda_n(\alpha) := \alpha_{n+1} + \frac{q_{n-1}}{q_n} = [a_{n+1}, a_{n+2}, ...] + [0, a_n, ..., a_1]$
Hence we have that

$$L(\alpha) = \lim_{n\to\infty} sup \lambda_n(\alpha)$$

2.2 The Markov Spectrum

A quadratic form is a function of the form

$$f(x,y) = ax^2 + bxy + cy^2$$

where $a, b, c \in \mathbf{R}$ and $(x,y) \in \mathbf{Z^2}$ and its discriminant is $\Delta(f) = b^2 - 4ac$. A quadratic form is called indefinite if $\Delta(f) > 0$ and definite if $\Delta(f) < 0$. Because $f(x,y) = a(x+\frac{b}{2a}y)^2 - \frac{\Delta(f)}{4a}y^2$, a definite form takes only nonnegative or nonpositive values, while an indefinite form takes on both positive and negative values. For an indefinite form f, define $m(f) = inf(|f(x,y)| : f(x,y) \neq 0, (x,y) \in \mathbf{Z^2})$ and define

$$M(f) = \frac{\sqrt{\Delta(f)}}{m(f)}$$

The Markoff Spectrum is the set of values of M(f) over all indefinite forms f. The Markoff Spectrum is actually also the set of values of

$$M(A) = sup \lambda_n(A)$$

which is tantalizingly similar to the previous

$$L(A) = \lim_{n\to\infty} sup \lambda_n(A)^1$$

3 Developing Markov Numbers

3.1 Markov's Equation

Markov's equation is the Diophantine equation

$$x_1^2 + x_2^2 + x_3^2 = 3x_1x_2x_3$$

The first triples of solutions are (1,1,1), (1,1,2), (1,2,5), (1,5,13), (2,5,29), (1,13,34), ... so the set of solutions starts $\mathcal{M} = \{1, 2, 5, 13, 29, 34, ...\}$

The triples of solutions for the Markov equation are coprime to one another and the equation

$$x_1^2 + x_2^2 + x_3^2 = kx_1x_2x_3, k \in \mathbf{N}$$

consequently only has solutions for k = 1 and k = 3.
All of the solutions can be found recursively through the rule $m_1' = 3m_2m_3 - m_1$ and can be compiled through the format of a tree.

[1] Here A is a doubly infinite sequence instead of a singly infinite sequence representing a continued fraction.

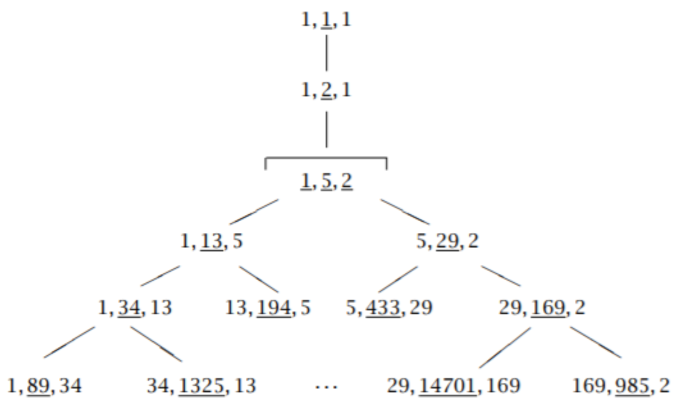

Figure 1: Markov Tree T_M [1]

$m^2 + m_2^2 + m_3^2 = 3mm_2m_3$, hence $m_2^2 \equiv -m_3^2(mod m)$. So $m_2x \equiv \pm m_3$ has two solutions between 0 and m, call them u and u'. Take v to be the integer such that $u^2 = -1 + mv$.

3.2 Markov's Theorem

The Langrange Spectrum and the Markov Spectrum below 3 are the exact same and are both:

$$\left\{ \frac{\sqrt{9m^2 - 4}}{m} : m \in \mathcal{M} \right\}$$

More accurately, Markov proved that there is a sequence of quadratic irrationals with these lagrange numbers and there are a series of quadratic forms with these markoff numbers and that for all irrationals and forms with corresponding values below 3, there is an equivalent irrational or form with the same Lagrange or Markov Number.

3.3 The Uniqueness Conjecture

Every Markov number appears exactly once as the maximum in a Markov triple.

Uniqueness Results:
Let $m \in \mathcal{M}, m \geq 5$. If $x^2 \equiv -1 mod m$ is uniquely solvable in (0, m/2), then m is unique.
Every Markov number m of the form $m = p^k$ or $m = 2p^k$ with p an odd prime is unique.
Every Markov number of the form $m = \frac{2^l p^k \pm 2}{3}$ with $l = 0, 1, 2, 3$ and p an odd prime is unique.

3.4 The Farey Tree

It is sometimes helpful in further developing these numbers to correspond the Markov Tree to the Farey Tree of reduced rationals in the [0,1] interval. We can see that starting with $\frac{0}{1}$ and $\frac{1}{1}$ we can find all rationals in [0,1] through the mediant operation $\frac{a}{b} \oplus \frac{c}{d} = \frac{a+c}{b+d}$. Below, the first four rows of the Farey table are depicted:

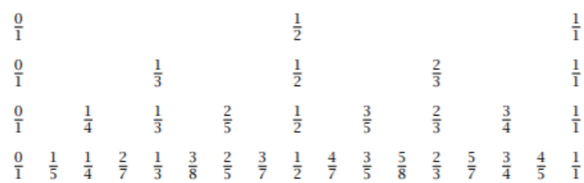

Figure 2: Farey Table [1]

We can use these Farey fractions to index our Markov numbers which will currently be marginally helpful but whose implications will be more obvious in the future.

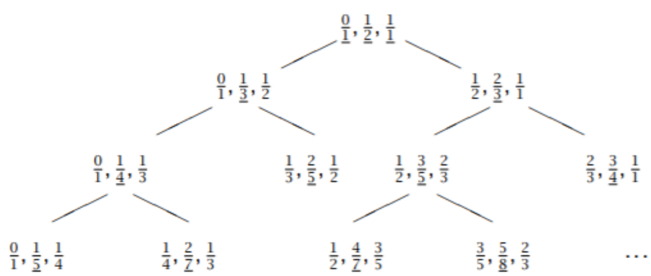

Figure 3: Farey Tree [1]

3.5 Cohn Matrices

$SL(2,\mathbf{Z}) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} : a,b,c,d \in \mathbf{Z}, ad-bc=1 \right\}$

A matrix $C = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in (2,\mathbf{Z})$ is a Cohn matrix if there is a Markov number $m \in \mathcal{M}$ such that $b = m, tr(C) = a + d = 3m$

All Cohn matrices representing the starting Markov triple (1,5,2) are given for $a \in \mathbf{Z}$ by

$$C_{\frac{0}{1}} = \begin{pmatrix} a & 1 \\ 3a-a^2-1 & 3-a \end{pmatrix}, C_{\frac{1}{1}} = \begin{pmatrix} 2a+1 & 2 \\ -2a^2+4a+2 & 5-2a \end{pmatrix}, C_{\frac{1}{2}} = \begin{pmatrix} 5a+2 & 5 \\ -5a^2+11a+5 & 13-5a \end{pmatrix}$$

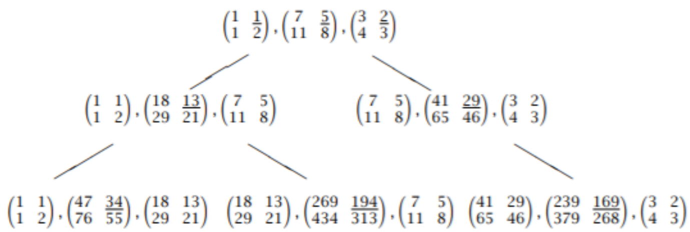

Figure 4: Cohn Tree [1]

3.6 Cohn Words

We can treat the Cohn matrices similarly by replacing matrix multiplication by concatenation of strings. This will result in the same triples only matrices will be replaced by longer and longer words:

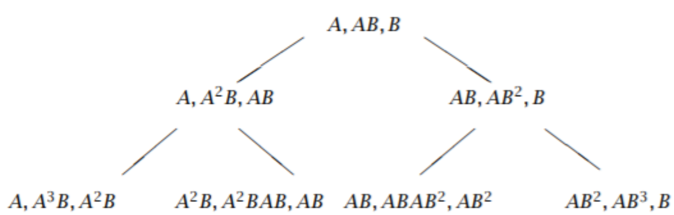

Figure 5: Word Tree [1]

We can define λ and ρ as the following members of automorphisms of the free group F(A,B):

$$\lambda : \begin{matrix} A \to A \\ B \to AB \end{matrix}, \quad \rho : \begin{matrix} A \to AB \\ B \to B \end{matrix}$$

Through some work we can actually see that the *set of positive automorphisms* $\{\lambda^{k_1}\rho^{l_1}...\lambda^{k_s}\rho^{l_s} : k_i, l_i \geq 0\}$ has a 1 to 1 correspondence with Cohn words.

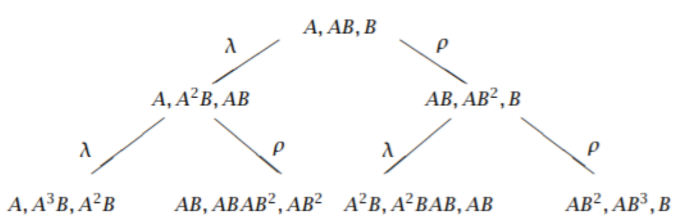

Figure 6: Word Tree Automorphisms [1]

3.7 Markov Words

We will now draw the correspondence between doubly infinite sequences and doubly infinite words. We say $A \leftrightarrow 11$ and $B \leftrightarrow 22$. We call a sequence strongly admissable and a word strongly Markov if its Lagrange number is strictly less than three. Through some work on infinite words(the study of Sturmian words and admissable sequences), we can eventually reduce our infinite word to see the following important result:

z is a strong Markov word if and only if $z = {}^{\infty}y^{\infty}$ where y is a Cohn word.

Through some further work on Christoffel words we are able to calculate the values on the spectra of these periodic words and determine the spectrum below 3.

t	W_t	y_t	$L(y_t)$
$\frac{0}{1}$	A	$\frac{1+\sqrt{5}}{2}$	$\sqrt{5} \approx 2.23607$
$\frac{1}{1}$	B	$1+\sqrt{2}$	$\sqrt{8} \approx 2.82843$
$\frac{1}{2}$	AB	$\frac{9+\sqrt{221}}{10}$	$\frac{\sqrt{221}}{5} \approx 2.97321$
$\frac{1}{3}$	A^2B	$\frac{23+\sqrt{1517}}{26}$	$\frac{\sqrt{1517}}{13} \approx 2.99605$
$\frac{2}{3}$	AB^2	$\frac{53+\sqrt{7565}}{58}$	$\frac{\sqrt{7565}}{29} \approx 2.99921$
$\frac{1}{4}$	A^3B	$\frac{15+\sqrt{650}}{17}$	$\frac{\sqrt{2600}}{17} \approx 2.99942$

Figure 7: First Lagrange Numbers [1]

4 More about the Spectra

Both the Lagrange Spectrum and the Markov Spectrum are closed sets and $\mathbf{L} \subset \mathbf{M}$.

If we denote $\mathbf{P}$ as M(A):A is a purely periodic doubly-infinite sequence and $\mathbf{Q}$ as M(A):A is an eventually periodic sequence (on both sides), then $\mathbf{L} = \mathrm{cl}(\mathbf{P})$ and $\mathbf{M} = \mathrm{cl}(\mathbf{Q})$.

We also know some of the gaps in the spectrum, but overall we don't have an entire understanding of either spectrum:

Both $(\sqrt{12}, \sqrt{13}) and (\sqrt{13}, \frac{1}{22}(9\sqrt{3}+65))$ are maximal gaps in the Markov spectrum.

4.1 Hall's Ray[3]

In 1947, Marshall Hall Jr. proved the following sequence to show that there is an infinite line segment contained in $\mathbf{L}$ and $\mathbf{M}$.

Any real number in the interval $[\sqrt{2}-1, 4\sqrt{2}-4]$ can be written in the form $[0; b_1, b_2, b_3, ...] + [0; c_1, c_2, c_3, ...]$ with b's and c's 4 or below.
Any real number can be written in the form $a + [0; b_1, b_2, b_3, ...] + [0; c_1, c_2, c_3, ...]$ with b's and c's 4 or below.
The Lagrange spectrum (and hence the Markov spectrum) contains $(6, \infty)$.

Take $\mu = 4 + [0; 3, 2, 1, 1, \overline{3, 1, 3, 1, 2, 1}] + [0; 4, 3, 2, 2, \overline{3, 1, 3, 1, 2, 1}] \approx 4.527829566$
Take $\nu = 4 + [0; 3, 1, 3, 1, 2, 1, 1, 3, 3, \overline{3, 1, 3, 1, 2, 1}] + [0; 3, 1, 3, 1, 3, \overline{4, 4, 4, 3, 2, 3}] \approx 4.527829538$
It took Freiman over 100 pages to show in 1975 that (ν, μ) is a gap in the Markov spectrum and that $[\mu, \infty)$ is contained in the Lagrange spectrum. Consequently,

$$\mu = 4 + \frac{253,589,820 + 283,748\sqrt{462}}{491,993,569}$$

is labeled Freiman's constant and is the beginning of the largest ray contained in both the spectra.

4.2 More about the Center of the Spectra [2]

For all $t \in \mathbf{R}$, $HD(\mathbf{L} \cap (-\infty, t)) = HD(\mathbf{M} \cap (-\infty, t))$
Moreover, $d(t) := HD(\mathbf{L} \cap (-\infty, t)) = HD(\mathbf{M} \cap (-\infty, t))$ is a continuous and surjective function from $\mathbf{R}$ to [0,1].
The set of accumulation of points of **L**, **L'**, is a perfect set, meaning **L"** = **L'**
Gugu has also more recently shown that $0.353 < HD(M - L) < 0.986927$.

References

[1] M. Aigner. *Markov's Theorem and 100 Years of the Uniqueness Conjecture.* Springer, 2013.

[2] Carlos Gustavo Moreira. Geometric properties of the markov and lagrange spectra. 2009.

[3] Mary E. Flahive Thomas W. Cusick. *The Markoff and Lagrange Spectra.* American Mathematical Society, 1989.