What is the Yang-Baxter Equation?

Will Hoffer

June 14, 2018

Abstract

The Yang-Baxter equation (YBE) is attributed to independent work of C. N. Yang and R. J. Baxter from the late 1960s and early 1970s. This equation gives an exchange relation which, in the context of knot theory, is tantamount to the third Reidemeister move equivalence. The YBE plays a pivotal role in the context of braid groups, serves as a bridge between statistical mechanics and knot theory, and much more. In this talk, we discuss the YBE as a braid group relation, in the context of abstract tensors arising from formal Feynman diagrams, and demonstrate solutions related to the Jones polynomial.

References

[1] Adams, Colin C. *The Knot Book*. American Mathematical Society, 2004.

[2] Kauffman, Louis H. *Knots and Physics*. World Scientific, 2001.

[3] Turaev, V.G. "The Yang-Baxter equation and invariants of links." Inventiones Mathematicae, 1988.

What is the Yang-Baxter Equation?

Common Presentation:

Let V be a vector space, and V^n its n^{th} tensor power.
Let $R: V^2 \to V^2$ be an invertible linear transform and $I: V \to V$ the identity map. Then the YBE is the following:

$$(R \otimes I)(R \otimes I)(R \otimes I) = (I \otimes R)(R \otimes I)(I \otimes R)$$

The YBE & Braid Groups

We introduce the notion of braid groups to demonstrate how the YBE encodes a Reidemeister exchange move.

Formal Def. of Braid:

Let $\mathbb{D}$ be the unit disk in $\mathbb{C}$, and consider n labeled points in $\mathbb{D} \cap \mathbb{R}$, $-1 < p_1 < p_2 < \dots < p_n < 1$. A braid is a collection of n paths, or strands, $f_i: [0,1] \to \mathbb{D}$ satisfying: for each $i \in \{1, 2, \dots, n\}$:

(1) $f_i(0) = p_i$

(2) $f_i(1) = p_{\tau(i)}$ for a fixed permutation $\tau \in S_n$

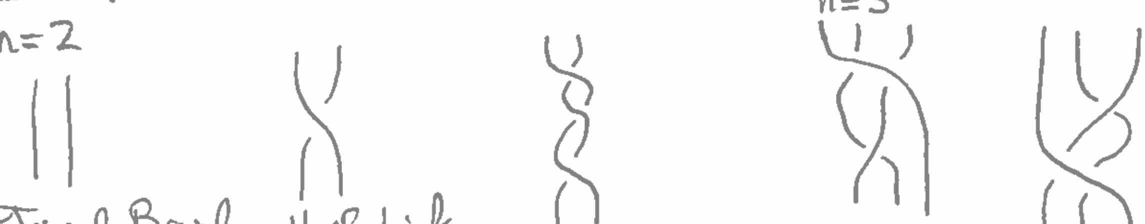
(3) $\forall t \in [0,1]$, if $i \neq j$, then $f_i(t) \neq f_j(t)$

Informally, a braid may be thought of a set of strands which are crossed over each other.

Examples

n=2

n=3



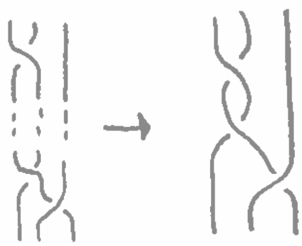
Trivial Braid   Hopf Link   Trefoil, 3_1 Reidemeister Move III

☆ This picture is visually the YBE.

NB: The knots/links are associated to the closure of these braids. Braid closure is connecting the corresponding top & bottom points together w/o

The Braid Group B_n

Braids form a group structure with the group operation of "glueing" two braids' corresponding strands together. (For the formal definition, one would appropriately piece together the n strand paths. It is particularly useful to depict two braids one above another and connecting points.)

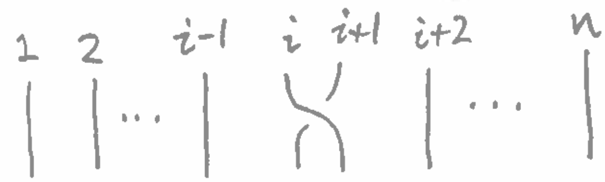


Combining two braids.

- The identity braid is the trivial braid.
- Associativity and closure are easily checked.
- The existence of inverses is apparent through considering the following simple braids.

For $i \in \{1, 2, \dots n-1\}$, consider the braids s_i, s_i^{-1}, given by:

s_i



s_i^{-1}

Note that evidently $s_i s_i^{-1} = 1 = s_i^{-1} s_i$, since the two interacting strands are equivalent under the Reidemeister move II:

Braids are unchanged under isotopy and the Reidemeister moves, so they may be decomposed into these simple transpositions. We formalize this notion in a moment, leaving the verification of the group structure to the audience.

(NB: The astute may note that inverses in the formal sense arise from reversing the parametrized path functions.)

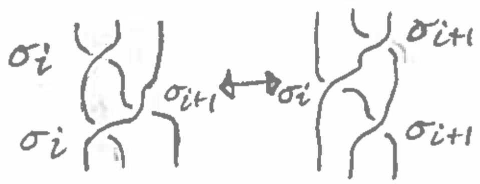
Artin's Presentation of the Braid Group

We construct $\mathfrak{B}_n$ as a set of words formed by the $n-1$ symbols $\sigma_1, \sigma_2, \ldots, \sigma_{n-1}$ and then take equivalence classes of words that are equivalent under two types of relations. Formally, we may define:

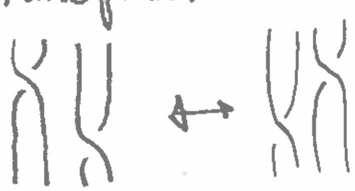
$$\mathfrak{B}_n := \langle \sigma_1, \sigma_2, \ldots, \sigma_{n-1} \mid \text{if } |i-j|>1,\ \sigma_i\sigma_j = \sigma_j\sigma_i, \\ \text{if } i=1,2,\ldots,n-2,\ \sigma_i\sigma_{i+1}\sigma_i = \sigma_{i+1}\sigma_i\sigma_{i+1} \rangle$$

for those familiar with group presentations. The symbols σ_i are the generators and σ_i's, with σ_i^{-1}'s, may be thought of as the symbols in the construction above, (with the extra relation that $\sigma_i\sigma_i^{-1} = 1 = \sigma_i^{-1}\sigma_i$, where 1 denotes the identity.)

NB: $\sigma_i\sigma_{i+1}\sigma_i = \sigma_{i+1}\sigma_i\sigma_{i+1}$ is the YBE.



Essentially, the YBE gives the crucial relation for this group. (The other relation simply states that non-adjacent transpositions commute.)



Theorem The braid group B_n and the group $\mathfrak{B}_n$ with the YBE relation due to Artin are isomorphic (as groups.)

Proof

Let $\phi: \mathcal{B}_n \to B_n$ be defined by the mapping of the σ_i's to their corresponding s_i braids.

ϕ can be seen to be a homomorphism since
(1) the YBE relation corresponds to the third Reidemeister move
(2) the non-adjacent commutation relation corresponds to an isotopy
(3) inverses in $\mathcal{B}_n$ correspond to the second Reidemeister move
(4) braids in B_n are equivalent up to isotopies

etc. The details follow from these structural preservations.

Showing that ϕ is an isomorphism also requires noting that it is both injective and surjective.

Surjectivity follows from noting that any braid in B_n (viz a pictoral braid) can be formed via individual transpositions (and isotopies), where $\phi(\sigma_i^{\pm 1})$'s are precisely these elements.

Injectivity can be shown by constructing a map ψ so that $\psi \circ \phi = id$ (the identity of $\mathcal{B}_n$)
(Given such a ψ, suppose $\exists x, y \in \mathcal{B}_n$, $x \neq y$ and $\phi(x) = \phi(y)$, i.e. ϕ is not injective. Then $x = \psi(\phi(x)) = \psi(\phi(y)) = y$, a contradiction.)
ψ may be constructed by using isotopies (which do not change the a braid in B_n) to express a braid in the form of the diagrams we have been using w/ crossings one at a time in a vertical progression. Take ψ to map a sequence of the form $\prod_\ell s_{i_\ell}^{\varepsilon_\ell}$, $\varepsilon_\ell = \pm 1$, to the word $\prod_\ell \sigma_{i}^{\varepsilon_\ell}$. ψ is well defined since isotopies and Reidemeister moves, which do not affect $b \in B_n$, are also respected by $\psi(b) \in \mathcal{B}_n$. Refer to the relations and their correspondences previously mentioned. (Details are left to the audience.)

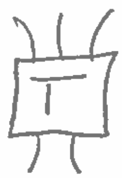
Kauffman's Take: The YBE & Abstract Tensors

One of the most interesting aspects of the YBE is in its connection between knot theory and physics. Now we present the notion of abstract tensors so as to show another form of the YBE and how it connects to Feynman diagrams, vacuum-vacuum expectations, and more complicated algebraic structures relevant to physics (eg. quantum groups.) We will have time to present how one may find solutions to the YBE in this context using a knot invariant, the Jones polynomial, motivated by the YBE's connection to knot theory that we have seen.

Abstract Tensor

$\leftrightarrow M = (M^i_j)$

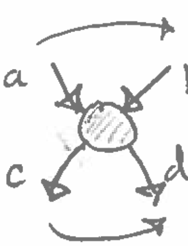
matrix M, entries M^i_j



$\leftrightarrow T^{ijk}_{lm}$

Tensor Object

We convert diagrams into tensor-like objects with some rules:



read inputs clockwise

read outputs counter-clockwise

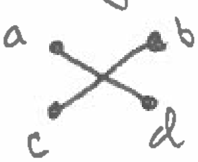
$\leftrightarrow S^{ab}_{cd}$

$\leftrightarrow \delta^b_a$ Kronecker delta


$\leftrightarrow M_{ab}$ annihilation matrix


$\leftrightarrow M^{ab}$ (sim. for creation matrix)

Crossing Convention:

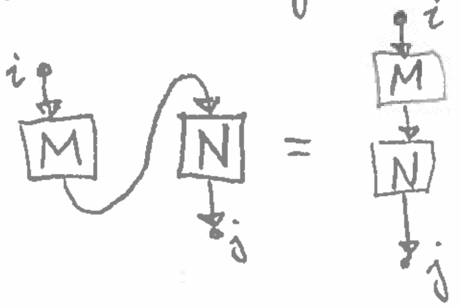


$\leftrightarrow \delta^a_d \delta^b_c$

Examples

Matrix Multiplication:

Einstein Summation Convention

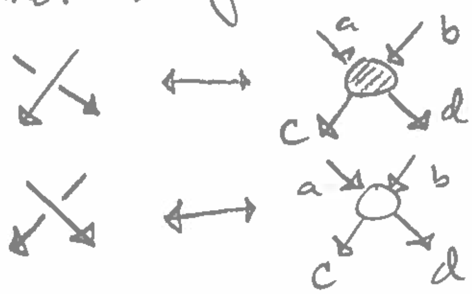


$$\longleftrightarrow \quad (MN)^i_j = \sum_k M^i_k N^k_j = M^i_k N^k_j$$

Trace:

$$= \sum M^i_i$$

Knot diagrams can be converted to such diagrams.



$$= R^{ab}_{cd}$$

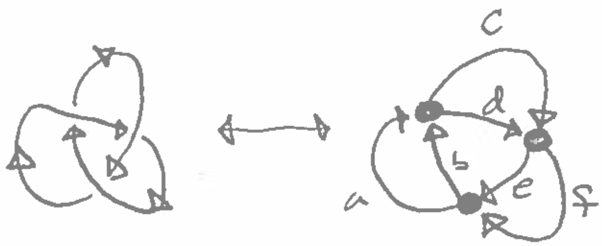
$$= \overline{R}^{ab}_{cd}$$

Oriented knot diagram K

$\Updownarrow$

Abstract tensor diagram $T(K)$

Eg: Trefoil (3_1)



3_1

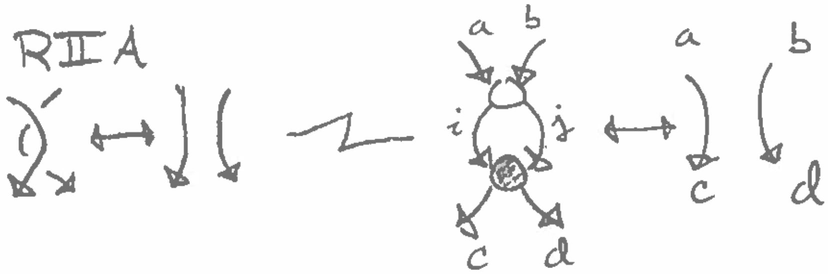
$$T(3_1) = R^{ba}_{dc} R^{dc}_{ef} R^{ef}_{ba}$$

Remark: The labeling of strands is a state, or formally a mapping from the edges/strands of the diagram into an index set $I = \{a, b, c, d, e, f, \dots\}$.

Reidemeister Moves as Diagrams

Channel Unitarity

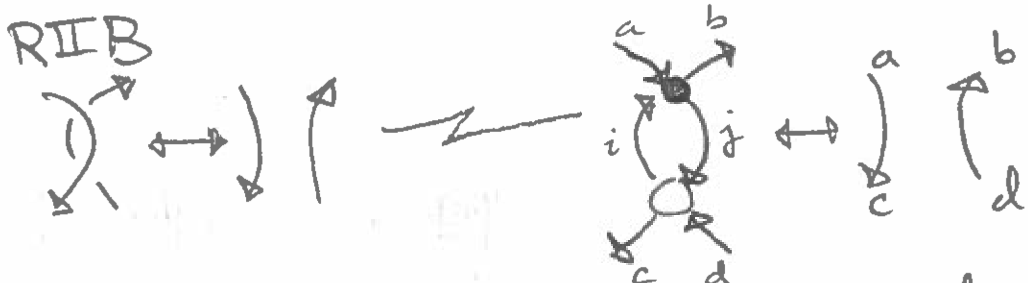
RII A



$$\overline{R}^{ab}_{ij} R^{ij}_{cd} = \delta^a_c \delta^b_d$$

Cross-Channel Unitarity

RII B

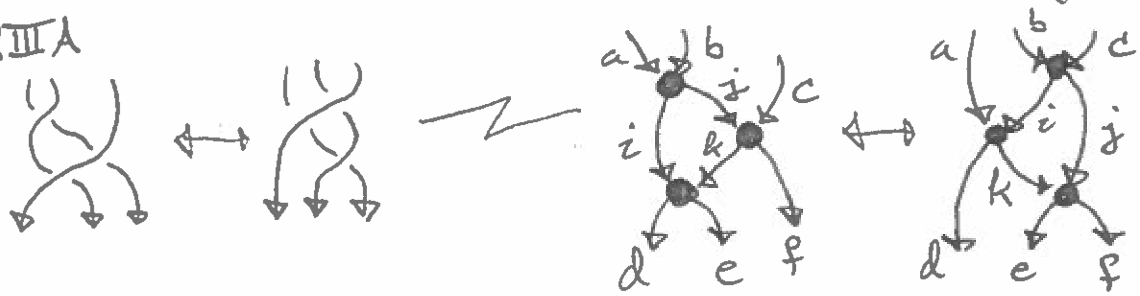


$$R^{ab}_{ij} \overline{R}^{jd}_{ic} = \delta^a_c \delta^d_b$$

NB: These oriented Reidemeister moves correspond to these two unitarities. In particular, these force R and $\overline{R}$ to be inverses.

Given the above, it suffices to have the following YBE:

RIII A



$$R^{ab}_{ij} R^{jc}_{kf} R^{ik}_{de} = R^{bc}_{ij} R^{ai}_{dk} R^{kj}_{ef}$$

YBE for R (sim. for $\overline{R}$)

With all of the above, $T(K)$ is a regular isotopy invariant for oriented knot diagrams K.

Finding a Solution to the YBE.

We construct a specific solution to the YBE using a construction from knot theory, the Jones polynomial.

Theorem Define $R^{ab}_{cd} = A\delta^a_c \delta^b_d + B\delta^{ab}\delta_{cd}$.

When $B(A^2 + nAB + B^2) = 0$, then R will satisfy the YBE. (Here, $n = -A^2 - A^{-2}$, or $nA^2 + A^4 + 1 = 0$)

Motivation: The Jones polynomial is constructed via splittings via skein relations, such as:

$$\langle \times \rangle = A \langle)\,(\rangle + B \langle \asymp \rangle$$

Compare this to the diagram corresponding to R's construction. We easily see that if $B = A^{-1}$, the theorem gives a solution to the YBE. This choice — and in fact, the choice of R and n — come directly from the corresponding choices made in constructing the Jones polynomial.

NB:
- $B = 0$ gives trivial solutions.
- We can more generally choose $n = -\left(\frac{A}{B} + \frac{B}{A}\right)$ (and we want $A \neq 0 \neq B$).

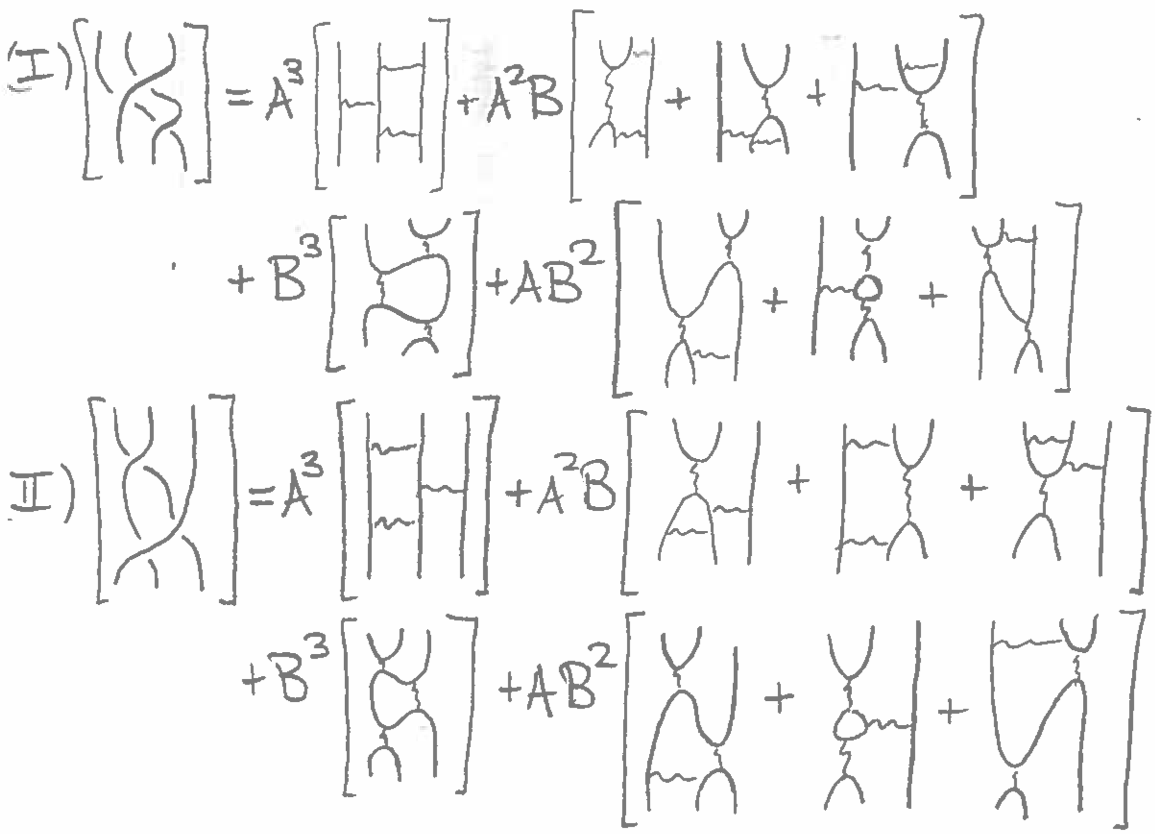
Proof

For the YBE to be satisfied, it suffices to show that the tensors assigned to the crossings will respect the third Reidemeister move.

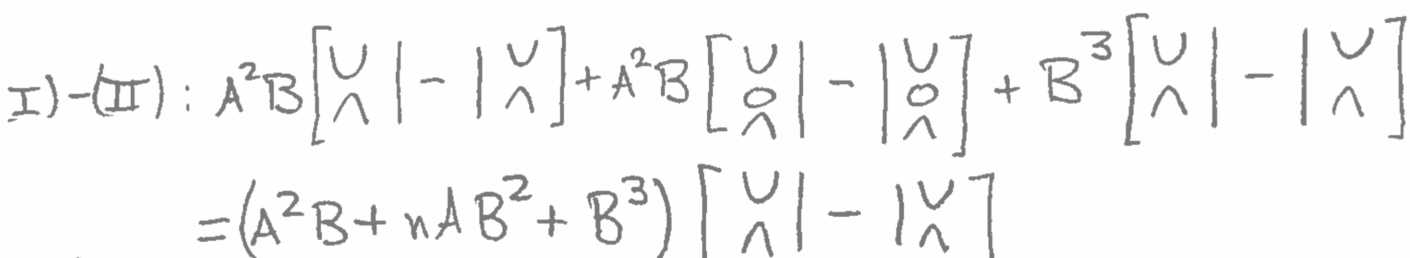
We use the fact that R is given as a decomposition of a crossing into two strand splittings to decompose the RIII move.

The proof can be presented pictorially, where we denote the splitting

(Compare: $R^{ab}_{cd} = A\delta^a_c\delta^b_d + B\delta^{ab}\delta_{cd}$)



Note: In crossing splittings, n is the value assigned to a disjoint circle. So, $[K \cup O] = n[K]$.



The YBE holds exactly when this coefficient is zero.

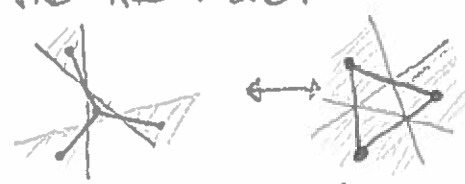
Supplemental Information

The YBE first appeared in the independent papers of C.N. Yang and R.J. Baxter in the late 1960's/early 1970's.

The YBE plays an important role in some exactly solvable models for statistical mechanics, namely the Pott's model. It is sometimes called the star-triangle relation, as it describes how a star shape in the model's lattice may be exchanged with a deletion (see image.)

"Star" Triangle

(Compare with the graphs associated to the RIII move:


)

For those with physics knowledge, this relation is satisfied by Boltzmann weights in the partition function for this to be an equivalent exchange.
When this condition holds, suddenly the partition function becomes RIII invariant, with a few tweaks, the entire partition function is a knot invariant.
In the Ising model (a special case of the Potts model w/ $q=2$), one can obtain the Arf invariant, and the Potts model version leads to the Jones polynomial $V(t)$ (where $q=2+t+t^{-1}$).