What is the Isoperimetric Inequality?

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Abstract

The isoperimetric inequality states the intuitive fact that, among all shapes with a given surface area, a sphere has the maximum volume. This talk explores a proof of this fact for subsets of $\mathbb{R}^n$ via the Brunn-Minkowski theorem.

1 Introduction

The isoperimetric ("same perimeter") inequality for $\mathbb{R}^n$ is stated as follows: For any subset $X \subset \mathbb{R}^n$ where it makes sense (we will explore this later), we have

$$\mathrm{Surf}(X) \geqslant \mu(X)^{\frac{n-1}{n}}\, \mu(B_1^n)^{\frac{1}{n}},$$

where μ gives the volume of a set, Surf gives its surface area, and B_1^n is the standard unit ball in $\mathbb{R}^n$.

The precise definitions of these concepts for more general sets X will be motivated by the proof and stated more precisely at the end of this document. For now though, assume that μ gives the standard notion of volume (say, Lebesgue measure) on sets where it makes sense. Further assume that for sets X where ∂X can be smoothly parameterized, $\mathrm{Surf}(X)$ agrees with the standard "cross product integral" definition of surface area from calculus in $\mathbb{R}^3$, and assume it agrees with the definition of arc length in for rectifiable curves in $\mathbb{R}^2$.

For illustration, here is the inequality for several fixed n:

Space	**Isoperimetric Inequality**
$\mathbb{R}^1$	$\#(\partial X) \geqslant 2$ for X with positive length.
$\mathbb{R}^2$	$\mathrm{Length}(\partial X) \geqslant 2\sqrt{\mathrm{Area}(S)}\sqrt{\pi}$
$\mathbb{R}^3$	$\mathrm{Area}(\partial X) \geqslant 3 \cdot \mu(S)^{2/3} \left(\frac{4\pi}{3}\right)^{1/3}$

Using general formulae for the volume and surface area of an n-ball, it is straightforward to verify that when X is such a ball, equality holds; this bound is tight.

The problem for $\mathbb{R}^2$ is ancient and closely related to Dido's Problem. In 1902, A. Hurwitz [3] proved the inequality for arbitrary rectifiable closed curves in the plane using Fourier series. E. Schmidt [7] gave a more direct proof using Green's

Theorem for the case of smooth closed curves in the plane and generalized to curves on other surfaces. These proofs are described in [6].

Instead of attacking this problem from the perspective of parameterized curves and the regions that they bound, the rest of this talk will explore a proof from Federer's foundational book [2], which defines volume and surface area for more general subsets of $\mathbb{R}^n$ without resorting to parameterization.

2 The Brunn-Minkowski Theorem

The major lemma for our proof will be the Brunn-Minkowski theorem, which was independently proven for convex sets by Brunn in 1887 [1] and Minkowski in 1896 [5]. In 1935, Lyusternik [4] generalized the theorem to compact sets.

We will work with the following general version: for Lebesgue-measurable $A, B \subset \mathbb{R}^n$, we have

$$\sqrt[n]{\mu(A \oplus B)} \geqslant \sqrt[n]{\mu(A)} + \sqrt[n]{\mu(B)},$$

Where $A \oplus B$ is the *Minkowski Sum* $\{a + b \mid a \in A,\, b \in B\}$. This sum has a few interesting properties:

- For $k \in \mathbb{R}^n$, we have $(A \oplus \{k\}) \oplus B = (A \oplus B) \oplus \{k\}$. In other words, a shift in one of the addends simply produces a shift in the sum. Because volume is shift-invariant, it suffices to prove the theorem for a shifted copy of either set.
- For the reason above, we can "add two pictures together" without knowing where the origins are and produce another picture where the origin is also not distinguished; up to translations, the sum is an affine property. See Figure 1 for some examples.
- Consider the Minkowski addition of boxes (Cartesian products of intervals):
$$\bigtimes_{i=1}^{n} [0, a_i) \oplus \bigtimes_{i=1}^{n} [0, b_i) = \bigtimes_{i=1}^{n} [0, a_i + b_i).$$
The second pictorial equation of Figure 1 gives an example.
- An equivalent statement of the Brunn-Minkowski theorem is that for all $\lambda \in [0, 1]$,
$$\sqrt[n]{\mu((1-\lambda)A \oplus \lambda B)} \geqslant (1-\lambda)\sqrt[n]{\mu(A)} + \lambda\sqrt[n]{\mu(B)},$$
where λX is the dilation $\{\lambda x \mid x \in X\}$. In other words, the function $X \mapsto \sqrt[n]{\mu(X)}$ is concave with respect to these "Minkowski convex combinations".
- **Exercise**: Provide some sufficient conditions on A and B so that equality holds in the Brunn-Minkowski theorem. Hint: when might some distributive properties hold?

- **Exercise**: Find $A, B \subset \mathbb{R}^2$ such that A and B are Lebesgue-measurable but $A \oplus B$ is not.

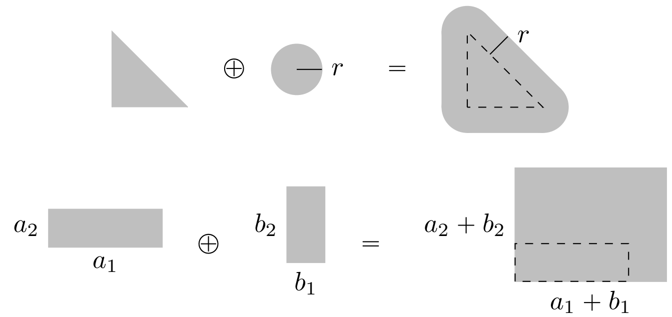

Figure 1: Minkowski sums of some simple subsets of $\mathbb{R}^2$

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We will now sketch a proof of the Brunn-Minkowski Theorem in three steps.

2.1 Proof for boxes

Suppose A and B are boxes (Cartesian products of intervals). As described above, without loss of generality, shift A and B so that

$$A = \mathop{\times}_{i=1}^{n} [0, a_i) \quad \text{and} \quad B = \mathop{\times}_{i=1}^{n} [0, b_i), \quad \text{and so} \quad A \oplus B = \mathop{\times}_{i=1}^{n} [0, a_i + b_i).$$

We then have

$$\mu(A) = \prod_{i=1}^{n} a_i, \qquad \mu(B) = \prod_{i=1}^{n} b_i, \quad \text{and} \quad \mu(A \oplus B) = \prod_{i=1}^{n} (a_i + b_i).$$

Using the AM-GM inequality, we have

$$\begin{aligned}
\frac{\sqrt[n]{\mu A} + \sqrt[n]{\mu B}}{\sqrt[n]{\mu(A \oplus B)}} &= \frac{\sqrt[n]{\prod a_i} + \sqrt[n]{\prod b_i}}{\sqrt[n]{\prod (a_i + b_i)}} \\
&= \sqrt[n]{\prod \frac{a_i}{a_i + b_i}} + \sqrt[n]{\prod \frac{b_i}{a_i + b_i}} \\
&\leqslant \frac{1}{n} \sum \frac{a_i}{a_i + b_i} + \frac{1}{n} \sum \frac{b_i}{a_i + b_i} \\
&= 1.
\end{aligned}$$

Multiplying by the initial denominator completes the proof for boxes.

2.2 Proof for finite collections of boxes

We now prove the theorem for when A and B are finite collections of disjoint half-open boxes: suppose that

$$A = \bigsqcup_{i=1}^{m_A} \bigtimes_{j=1}^{n} [a_{i,j,1},\, a_{i,j,2}) \qquad \text{and} \qquad B = \bigsqcup_{i=1}^{m_B} \bigtimes_{j=1}^{n} [b_{i,j,1},\, b_{i,j,2}).$$

We proceed by induction on $m_A + m_B$, the total number of boxes in either set. The base case of $m_A = m_B = 1$ is the previous part of the proof. Without loss of generality, A has at least two boxes $R_1 = \bigtimes_{j=1}^{n}[a_{k_1,j,1},\, a_{k_1,j,2})$ and $R_2 = \bigtimes_{j=1}^{n}[a_{k_2,j,1},\, a_{k_1,j,2})$. The boxes are disjoint, so there must be some half-space of the form $H = \{x \in \mathbb{R}^n \mid x_i \geqslant \lambda\}$ that separates them—such that $A \subset H$ and $B \subset \mathbb{R}^n \backslash H$ (or vice versa).

Replace A by $A \oplus \{-\lambda\}$, so that $H = \{x \in \mathbb{R}^n \mid x_i \geqslant 0\}$. Define $A^+ = A \cap H$, $A^- = A\backslash H$, $B^+ = B \cap H$, and $B^- = B\backslash H$. Replace B by $B \oplus \{\xi\}$ in such a way that we have

$$\frac{\mu\, A^+}{\mu\, A} = \frac{\mu\, B^+}{\mu\, B},$$

that is, translate B until the amounts of B on either side of the hyperplane ∂H are proportional to those amounts of A on either side.

We can now apply the inductive hypothesis on $A^+ \oplus B^+$ because each box in A or B contributed at most one box to A^+ or B^+, and there was one box explicitly excluded from A^+. The same holds for $A^- \oplus B^-$. Note that $A^+ \oplus B^+$ and $A^- \oplus B^-$ are disjoint because only the first has any $x_i \geqslant 0$. Now we have

$$\begin{aligned}
&\mu(A \oplus B) \\
&\geqslant \mu(A^+ \oplus B^+) + \mu(A^- \oplus B^-) && \text{disjoint subsets} \\
&\geqslant \left(\sqrt[n]{\mu\, A^+} + \sqrt[n]{\mu\, B^+}\right)^n + \left(\sqrt[n]{\mu\, A^-} + \sqrt[n]{\mu\, B^-}\right)^n && \text{induction hypothesis} \\
&= \mu(A^+)\left(1 + \sqrt[n]{\frac{\mu\, B^+}{\mu\, A^+}}\right)^n + \mu(A^-)\left(1 + \sqrt[n]{\frac{\mu\, B^-}{\mu\, A^-}}\right)^n \\
&= \left(\mu\, A^+ + \mu\, A^-\right)\left(1 + \sqrt[n]{\frac{\mu\, B}{\mu\, A}}\right)^n && \text{proportionality} \\
&= \left(\sqrt[n]{\mu\, A} + \sqrt[n]{\mu\, B}\right)^n.
\end{aligned}$$

This completes the induction for finite collections of boxes.

2.3 Generalization

Here I will gloss over the precise measure-theoretic proofs of the generalizations of the theorem to account for more general sets:

- We can generalize to *countable* disjoint unions of half-open boxes because $(A, B) \mapsto \sqrt[n]{\mu(A \oplus B)}$ is "continuous" with respect to the inclusion of more small boxes.

- **Exercise**: Show that we can write any open set in $\mathbb{R}^n$ as a countable union of disjoint half-open boxes. Conclude that Brunn-Minkowski holds for any open sets A and B.

- Federer [2] gives the following stronger generalization: for *any arbitrary subsets* $A, B \subseteq \mathbb{R}^n$, if we interpret μ as the *outer* Lebesgue measure, then the theorem still holds.

3 Final Proof: Applying Brunn-Minkowski

Let $X \subset \mathbb{R}^n$. We prove the isoperimetric inequality for X by applying Brunn-Minkowski to X and B_ε, the ball in $\mathbb{R}^n$ with radius ε, giving

$$\sqrt[n]{\mu(X \oplus B_\varepsilon)} \geqslant \sqrt[n]{\mu\, X} + \sqrt[n]{\mu\, B_\varepsilon}.$$

Because volume scales with the nth power of ε, the second term is $\varepsilon \cdot \sqrt[n]{\mu\, B_1}$. We now raise both sides to the nth power, giving

$$\mu(X \oplus B_\varepsilon) \geqslant \sum_{k=0}^{n} \binom{n}{k} \varepsilon^k\, \mu(B_1)^{k/n}\, \mu(X)^{(n-k)/n}.$$

We can now move the $k = 0$ term to the left side and divide everything by ε:

$$\frac{\mu(X \oplus B_\varepsilon) - \mu(X)}{\varepsilon} \geqslant \frac{1}{\varepsilon} \sum_{k=1}^{n} \binom{n}{k} \varepsilon^k\, \mu(B_1)^{k/n}\, \mu(X)^{(n-k)/n}.$$

Taking the limit, all terms vanish other than the $k = 1$ term:

$$\lim_{\varepsilon \downarrow 0} \frac{\mu(X \oplus B_\varepsilon) - \mu(X)}{\varepsilon} \geqslant n \cdot \mu(B_1)^{\frac{1}{n}}\, \mu(X)^{\frac{n-1}{n}}.$$

If we accept the definition

$$\mathrm{Surf}(X) = \lim_{\varepsilon \downarrow 0} \frac{\mu(X \oplus B_\varepsilon) - \mu(X)}{\varepsilon} = \lim_{\varepsilon \downarrow 0} \frac{\mu((X \oplus B_\varepsilon) \backslash X)}{\varepsilon},$$

then our proof of the isoperimetric inequality is complete. Intuitively, this quantity represents painting X with and ε-thick layer of paint and dividing the volume of paint added by its thickness to get the surface area. The first pictorial equation of Figure 1 gives an example of such "painting." When $\mu(X)$ is finite and this limit exists (or even when the lim sup is finite), the inequality is valid.

4 Precise Stronger Statement

This section provides a precise formulation of the strongest version of isoperimetric inequality proven in [2], for those curious. It uses a slightly different notion of surface area, and it can be considered supplementary.

We define the *Minkowski content* $\mathcal{M}$ of a set as follows:

$$\mathcal{M}(S) = \lim_{r \downarrow 0} \frac{\mu\{x : d(S,x) < r\}}{2}.$$

We also define $\mathcal{M}^*(S)$ and $\mathcal{M}_*(S)$ as the corresponding $\limsup$ and $\liminf$, respectively. Some remarks:

- For open sets X with smoothly parameterized ∂X, we have $\mathrm{Surf}(X) = \mathcal{M}(\partial X)$. Note that $\mathcal{M}(\partial X)$ counts "both sides" of a surface and then divides by 2, while $\mathrm{Surf}(X)$ counts only the outside.
- The Minkowski content is equal to the $(n-1)$-dimensional Hausdorff measure for rectifiable surfaces, and thus agrees with standard definitions of surface area for smooth surfaces.

The precise statement of the isoperimetric inequality that Federer proves is as follows: for $X \subset \mathbb{R}^n$ with $\mu\left(\overline{X}\right) < \infty$, we have

$$\mathcal{M}_*(\partial X) \geqslant n \cdot \mu(B_1)^{\frac{1}{n}} \cdot \mu\left(\overline{X}\right)^{\frac{n-1}{n}},$$

where $\overline{X}$ is the closure of X. Note the strengthening with both the $\liminf$ and the closure.

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