# What is the Rado Graph?

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**Abstract**

In 1963, Erdős and Rényi investigated what properties could be expected of a random infinite graph on countably many vertices. As it turns out, the theory of such graphs is remarkably easy: the same graph $R$ will arise with probability 1. This graph admits many more deterministic constructions and has some nice universality properties, lending the theory of infinite random graphs some beauty for what it lacks in variety.

## 1 Defining the Graph

We check out several ways in which the Rado graph $R$ was defined. I cannot say with certainty why Rado gets singled out; he came after both Ackerman and Erdős and Rényi. Rado's construction may have been glorified by Cameron in [2].

### 1.1 Binary Definition

This construction was given by Ackerman in [1] and Rado in [4] using the "BIT predicate", a statement which, given positive integers $x < y$, returns true when the $x$-th bit of $y$ is nonzero. The Rado graph then arises by taking $\{1, 2, \ldots\}$ as the set of vertices and joining vertices satisfying the BIT predicate. We can equivalently say that a vertex $n$ has an edge to any vertex congruent to any of $2^{n-1}, \ldots, 2^n - 1 \pmod{2^n}$.

### 1.2 Set Theoretic Definition

This definition of $R$ was also given by Ackermann in [1]. Don't get too upset by my lack of set theory knowledge; I don't think anything too deep is happening here on the set theory

front. Let $\mathcal{M}$ be a universe of hereditarily finite sets, starting with $\varnothing$. Join sets $x$ and $y$ by an edge when $x \in y$ or $y \in x$. Bam. There's $R$.

### 1.3 Probabilistic Definition

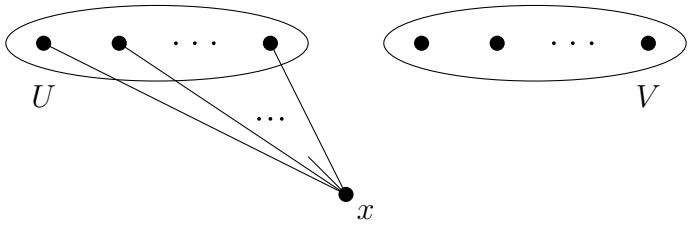
Erdős and Rényi gave probabilistic definition of $R$ in [3], opting not to provide any further construction and settling only for existence. They simply connected each pair of vertices by an edge independently with probability $\frac{1}{2}$.

### 1.4 Number Theoretic Definition

Take as our base set $P$ the set of primes congruent to 1 modulo 4. Join $p, q \in P$ by an edge when $(\frac{p}{q}) = 1$, namely when $p$ is a quadratic residue modulo $q$; quadratic reciprocity tells us that this happens precisely when $(\frac{q}{p}) = 1$. I have not seen anyone give this construction before Cameron in [2].

## 2 The Extension Property

We would like to verify that we have constructed the same graph in all of these cases (and only a single graph in the probabilistic case). It seems that the easiest way to do this is to verify that each graph possesses the *extension property*: given $U, V \subset R$ disjoint finite collections of vertices, there exists a vertex $x \in R - (U \cup V)$ which has an edge to each vertex of $U$ and no edges to vertices in $V$ (I may call $x$ a *witness* to extension for $(U, V)$ at times).




We care nothing for edges within or between $U$ and $V$, only that they share no vertices. The extension property is useful because of the following statement:

**Theorem 1.** *Let $\Gamma$ and $\Gamma^*$ be countable graphs satisfying the extension property. Then $\Gamma$ and $\Gamma^*$ are isomorphic.*

*Proof.* We define our isomorphism $f$ inductively. Let $\{x_1, x_2, \ldots\}$ and $\{y_1, y_2, \ldots\}$ be the vertex sets of $\Gamma$ and $\Gamma^*$ respectively. Let $f_0 = \varnothing$ be the empty map. Suppose we have constructed $f_n$, which is an isomorphism of some induced subgraphs of $\Gamma$ and $\Gamma^*$. We use a model-theoretic technique known as "back-and-forth" to extend $f_n$ to $\Gamma$. This involves artificially introducing cases. The conditions for the cases are simply things that will happen infinitely many times; the conditions themselves have no bearing on the proof.

*"Case" 1: ($n$ is even).* Let $m$ be the smallest index such that $f_n$ is not defined on $x_{m+1}$. Define $U, V \subset \text{domain}(f_n)$ by letting $U$ be the neighbors of $x_{m+1}$ and $V$ the non-neighbors. Since $\Gamma^*$ satisfies the extension property, there exists $y \in \Gamma^*$ which is adjacent to each vertex of $f_n(U)$ and no vertex in $f_n(V)$, so we may take such a $y$ to be $f_{n+1}(x_{m+1})$ (no need for Axiom of Choice, thanks to well-ordering).

*"Case" 2: ($n$ is odd).* Let $m$ be the smallest index such that $y_{m+1}$ is not in the range of $f_n$. We may repeat the earlier argument backwards. Define $U, V \subset \text{range}(f_n)$ by letting $U$ be the neighbors of $y_{m+1}$ and $V$ the non-neighbors. Now there exists $x \in \Gamma$ adjacent to each vertex of $f_n^{-1}(U)$ and no vertex in $f_n^{-1}(V)$, and we may take $f_{n+1}(x) = y_{m+1}$.

Now take $f = \bigcup_{n \geq 1} f_n$. This alternating method of defining $f$ has ensured that $f$ is 1-to-1 and onto; "case" 1 controls how we extend the domain while "case" 2 controls how we extend the range, so we do each case infinitely many times to control both. We know $f$ is a graph isomorphism because each $f_n$ is; check any edge to make sure it's preserved by looking at a large enough $n$. □

## 2.1 Verifying the Extension Property

By verifying each of our earlier characterizations of $R$ has the extension property, we will know that all are isomorphic by our theorem. Most of these verifications are really quick, so you can do them yourself or spoil the solutions by reading the rest of this section.

The binary definition satisfies extension almost immediately: given $U = \{m_1, \ldots, m_k\}$ and $V = \{n_1, \ldots, n_l\}$, take $x = \sum_{i=1}^{k} 2^{m_k}$, the number with 1's in the spots for $U$.

The probabilistic case is quick: given $U$ and $V$ disjoint and finite, each vertex outside $U \cup V$ has an independent $1/2^{|U|+|V|}$ chance of witnessing the extension property for $(U, V)$, so one of our infinitely many vertices will surely do it.

The set theoretic $R$ has the extension property with just a bit of set theory (shocking!) in the form of the Axiom of Foundation. Let $U = \{u_1, \ldots, u_m\}$ and $V = \{v_1, \ldots, v_n\}$. We can see that $x = U \cup \{V\}$ witnesses the extension property for $(U, V)$. We know that every $u \in U$ is adjacent to $x$, since $U \subseteq x$. Suppose for a contradiction that we have $v \in V$

adjacent to $x$. If $v \in x$, then since $v \neq u$ for all $u \in U$, we must have $v = V$, in which case $v \in V = v$, so $v \in v$, contradicting the Axiom of Foundation. If $x \in v$, then we note that $v \in V \in x$ to get $x \in v \in V \in x$, contradicting the Axiom of Foundation. Hence $x$ witnesses extension for $(U, V)$.

Let use check that our number theoretic construction is good. Let $U, V \subset P$ be disjoint finite subsets of the collection of primes congruent to 1 modulo 4. Write $U = \{u_1, \ldots, u_m\}$ and $V = \{v_1, \ldots, v_n\}$. Let $d = 4u_1 \cdots u_m v_1 \cdots v_n$. For each $i = 1, \ldots, m$, let $a_i$ be a quadratic residue modulo $u_i$, and for each $j = 1, \ldots, n$, let $b_j$ be a quadratic non-residue modulo $v_j$. The system

$$y \equiv 1 \pmod 4, \qquad y \equiv a_i \pmod{u_i}, \qquad y \equiv b_j \pmod{b_j}$$

where $i = 1, \ldots, m$ and $j = 1, \ldots, n$ has a unique solution $y \equiv z \pmod d$ by the Chinese Remainder Theorem. Since $a_1, \ldots, a_m, b_1, \ldots, b_n \neq 0$, we have that $z$ is not a multiple of any of $a_1, \ldots, a_m, b_1, \ldots, b_n$, so $z$ is coprime to $d$, so the arithmetic progression $z, z+d, z+2d, \cdots$ hits infinitely many primes by Dirichlet's Theorem. Take $x \in P$ to be one such prime, and $x$ will witness the extension property for $(U, V)$.

# 3 Universality

## 3.1 Induced Subgraphs

**Theorem 2.** *Every finite or countable graph arises as an induced subgraph of $R$.*

*Proof.* Let $\Gamma$ be our favorite finite or countable graph, with vertex set $\{v_1, v_2, \ldots\}$. We define an embedding $f : \Gamma \to R$ inductively much as in the extension proof, starting with $f_0 = \varnothing$. Suppose we have $f : \{v_1, \ldots, v_n\} \to R$ an isomorphism of induced subgraphs in $\Gamma$ and $R$. Let $U$ be the neighbors of $v_{n+1}$ in $v_1, \ldots, v_n$, and $V$ the non-neighbors. Choose $x$ in $R$ to have neighbors $f_n(U)$ and non-neighbors $f_n(V)$ and define $f_{n+1}(v_{n+1}) = x$ to extend our map. Taking $f = \bigcup_n f_n$ again gives our isomorphism, stopping the process if $\Gamma$ is finite. □

Some further results can be shown to get that $R$ has a partition into Hamiltonian paths. $R$ has some additional nice properties regarding partitions as well.

## 3.2 Partition Regularity

We set things up by showing a few nice stability results for $R$.

**Proposition 1.** *Let $U, V \subset R$ be finite disjoint subsets of vertices in $R$. Let $Z$ be the set of vertices which witness the extension property for $(U, V)$. Then $Z$ induces a subgraph isomorphic to $R$.*

*Proof.* We verify the extension property for the subgraph $\Gamma$ induced by $Z$. Let $U', V' \subset Z$ be finite and disjoint. Let $x \in Z$ be adjacent to all of $U \cup U'$ and none of $V \cup V'$. Then $x$ witnesses the extension property for $(U', V')$ in $\Gamma$. □

We will refer to the graph operation of *switching* with respect to a set $X$ of vertices, namely flipping all edges and non-edges going between $X$ and its complement. I will also refer to *flipping* an edge or non-edge to mean turning an edge into a non-edge or vice-versa.

**Proposition 2.** *The isomorphism type of $R$ is unchanged by an application of any of the following operations:*

*(a) Deleting a vertex*

*(b) Flipping an edge or non-edge*

*(c) Switching with respect to a finite set of vertices*

*Proof.* We show that the extension property is preserved. For (a), our only worry is that we delete our witness to the extension property; by proposition 1, there are infinitely many witnesses, so no problem arises. For (b), we worry that we have tampered with an edge for our witness; since there are infinitely many witnesses, pick one which does not involve the flipped (non-)edge. Lastly, for (c), let $X$ be our switching set and $U, V$ our disjoint finite collections of vertices. Pick $x$ to be a witness to $\big((U - X) \cup (V \cap X), (V - X) \cup (U \cap X)\big)$ to correct for the switching. □

Now we are ready to verify the partition-regular nature of $R$, also called the *pigeonhole principle* in [2].

**Theorem 3.** *For any finite partition of the vertices of $R$, the induced subgraph of one cell of the partition is isomorphic to $R$.*

*Proof.* Suppose the partition $C_1, \ldots, C_n$ provides a counterexample. Then no cell $C_i$ has the extension property; let $U_i, V_i$ be a counterexample to extension in each $C_i$. But then $U = U_1 \cup \cdots \cup U_n$ and $V = V_1 \cup \cdots \cup V_n$ is a counterexample to extension in $R$, a contradiction. □

The Rado Graph is also universal with respect to this property:

**Theorem 4.** *The only countable partition-regular graphs are the complete graph, the null graph, and R. Namely, only these three graphs are such that any finite vertex coloring yields a color whose induced subgraph is isomorphic to the original graph.*

*Proof.* Let $\Gamma$ be partition-regular, but not null or complete. Then $\Gamma$ has no isolated vertices and no vertices which neighbor all other vertices (as these can be partitioned out, then partition regularity can be applied). Suppose for a contradiction that $\Gamma$ is not isomorphic to $R$. Let $U = \{u_1, \ldots, u_m\}$ and $V = \{v_1, \ldots, v_n\}$ have no extension witness in $\Gamma$, with $m + n$ minimal among such counterexamples. We know $m + n > 1$ by our WLOG statement, we may partition $U \cup V$ into two nonempty subsets $A$ and $B$ (one of $U$ and $V$ may have been empty to start). Define $X$ to be all vertices in $A$ as well as all non-witnesses (not in $B$) to $(U \cap A, V \cap A)$. Define $Y$ to be all vertices in $B$ as well as all additional non-witnesses (not in $X$) to $(U \cap B, V \cap B)$. Now $X \cup Y$ is a partition of the vertices of $\Gamma$, since it fails the extension property, so one cell, WLOG $X$, induces a subgraph ismorphic to $\Gamma$. We have produced a smaller pair $(U \cap A, V \cap A)$ which has no extension witness if $\Gamma$, a contradiction. Hence $\Gamma$ is isomorphic to $R$. □

We also see that $R$ is self-complementary by noting that extension property is self-complementary, as is our probabilistic construction of $R$. In 1996, Pouzet and Sauer got a similar result for finite partitions of the edges of $R$: given a finite edge coloring of $R$, there is a subgraph of $R$ that is isomorphic to $R$ and only uses two colors of edges. One color may not be enough.

# 4 The Automorphism Group

Fewer proofs will be given here, as we will opt for an overview of a few nice properties about $\mathrm{Aut}(R)$. First, $\mathrm{Aut}(R)$ acts transitively on any finite configuration in $R$. We also know that $|\mathrm{Aut}(R)| = 2^{\aleph_0}$, and in fact has at least $2^{\aleph_0}$ elements which are not conjugate. Further, $\mathrm{Aut}(R)$ is simple, satisfying a stronger result shown by Truss: given $g, h \in \mathrm{Aut}(R)$, we may express $h$ as a product of five conjugates of $g$ or $g^{-1}$.

Lastly, we give one more universality property. Given a countable group $G$, we may construct a *random Cayley graph* by choosing whether or not to include the pair $x, x^{-1}$ in the generating with probability $\frac{1}{2}$. Someone optimistic may hope that we get $R$ almost surely, but this is not quite true for all countable groups. Necessary and sufficient conditions are known, but they are...gross. We'll just throw this result here and call it good:

Given a group $G$ and $g \in G$, write $\sqrt{g} = \{x \in G : x^2 = g\}$. We get the following:

**Theorem 5.** *Let $G$ be a countable group which cannot be covered by finitely many translates of sets $\sqrt{g}$ (where $g \in G$ with $g \neq 1$) and a finite set. Then almost all random Cayley graphs of $G$ are isomorphic to $R$.*

# 5 Other Stuff

Let me just say, set theorists and model theorists love $R$, because it is a first-order model of almost all finite graphs, shown by Fagin in 1976. Hence, many first-order logical statements about almost all finite graphs may be proved via $R$. Many other model-theoretic words can be used to describe $R$, and model theory may be the main site for applications of $R$ at this point.

To finish things off, let us point out a shortcoming of $R$ in the world of universality (in addition to the Cayley graph thing). We can quickly see that $R$ is not universal for isometric embeddings of finite graphs, as it has diameter 2. Lawrence Moss in 1989 and 1991 found universal graphs for isometric embeddings of graphs of fixed diameter, and $R$ is universal for diameter 2.

# References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