The Theorems of Ceva and Menelaus

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Outline

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1 Introduction

In their most basic form, Ceva's Theorem and Menelaus's Theorem are simple formulas of triangle geometry. To state them, we require some definitions.

- Three or more line segments in the plane are *concurrent* if they have a common point of intersection.

[1]The reverse is true, but complicated. See [Sil00] for more details.

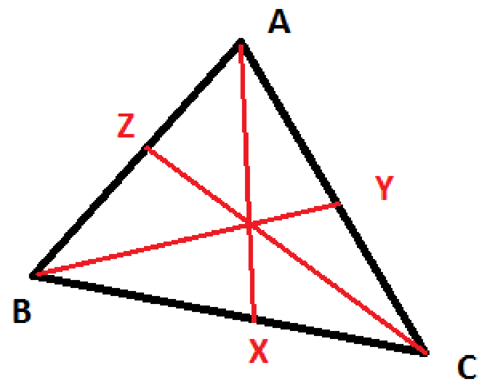

Figure 1: The basic case of Ceva's Theorem

- A *cevian* of a triangle ΔABC is a line segment with one endpoint at one vertex of the triangle (say A) and one endpoint on the opposite line (say $\overleftrightarrow{BC}$), but not passing through the opposite vertices (B or C).

We also denote the length of line segment $\overline{AB}$ to be $|AB|$.

Theorem 1.1 (Ceva's Theorem, Basic Version)**.** *Choose X on the line segment $\overline{BC}$, Y on the (interior of the) line segment $\overline{AC}$, and Z on the (interior of the) line segment $\overline{AB}$.*

Ceva's Theorem *If the cevians $\overline{AX}$, $\overline{BY}$, and $\overline{CZ}$ are concurrent, then*

$$\frac{|BX|}{|XC|} \cdot \frac{|CY|}{|YA|} \cdot \frac{|AZ|}{|ZB|} = 1. \tag{1.1}$$

Converse *If the points X, Y, Z are chosen as above, and if*

$$\frac{|BX|}{|XC|} \cdot \frac{|CY|}{|YA|} \cdot \frac{|AZ|}{|ZB|} = 1,$$

then the cevians $\overline{AX}$, $\overline{BY}$, and $\overline{CZ}$ are concurrent.

See Figure 1.

Theorem 1.2 (Menelaus's Theorem, Basic Version)**.** *Choose X on the line $\overleftrightarrow{BC}$ but not on the segment $\overline{BC}$, choose Y on the (interior of the) line segment $\overline{AC}$, and Z on the (interior of the) line segment $\overline{AB}$.*

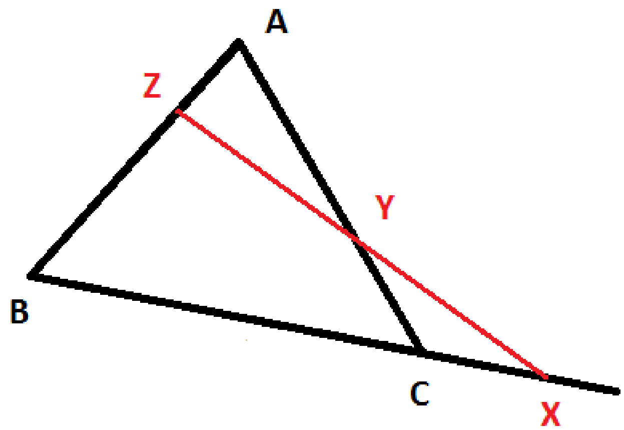

Figure 2: The basic case of Menelaus's Theorem

Menelaus's Theorem *If X, Y, and Z are collinear, then*

$$\frac{|BX|}{|CX|} \cdot \frac{|CY|}{|AY|} \cdot \frac{|AZ|}{|BZ|} = 1. \tag{1.2}$$

Converse *If the points X, Y, Z are chosen as above, and if*

$$\frac{|BX|}{|CX|} \cdot \frac{|CY|}{|AY|} \cdot \frac{|AZ|}{|BZ|} = 1,$$

then X, Y, and Z are collinear.

See Figure 1.

Menelaus's Theorem was known to the ancient Greeks, including Menelaus of Alexandria: a proof comes from Menelaus's *Spherica* ([OR99]). We have no evidence, however, that Ceva's theorem was discovered formally before Ceva's publication of *De Lineas Rectis* in 1678 ([OR12]). Nevertheless, the theorems have a certain similarity.

In fact, not to put too fine a point on it, except for the placement of X, the equations (1.1) and (1.2) look alike. What is going on here?

The general idea is that ultimately, Ceva's and Menelaus's Theorems are theorems about *signed* lengths ([Bog99], [Sil01]). If, for example, B, X, and C are collinear, then $\dfrac{|BX|}{|XC|}$ should be considered positive if X is between B and C (i.e., $\overrightarrow{BX}$ and $\overrightarrow{XC}$ are in the same direction) and negative otherwise (i.e, $\overrightarrow{BX}$ and $\overrightarrow{XC}$ are in opposite directions). To

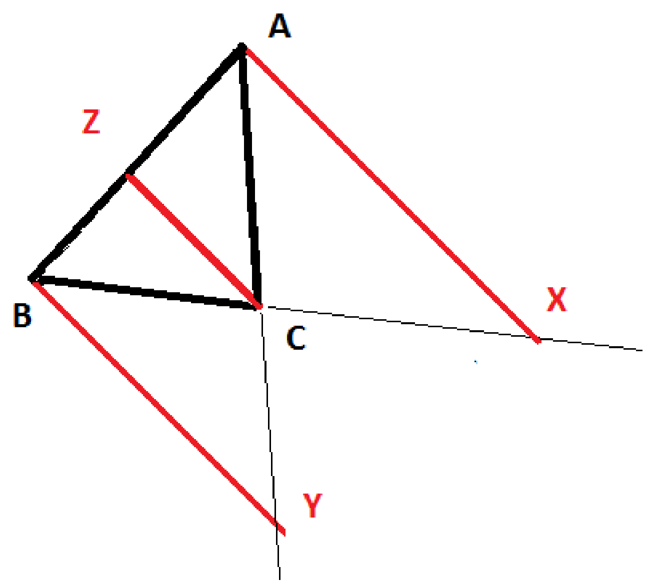

Figure 3: Another case of Ceva's Theorem

make this "signed length" clear, we will use the symbol $\frac{\left|\overrightarrow{BX}\right|}{\left|\overrightarrow{XC}\right|}$, and it will give the number λ such that the *vectors* $\overrightarrow{BX}$, $\overrightarrow{XC}$ satisfy $\overrightarrow{BX} = \lambda \cdot \overrightarrow{XC}$. Then in Menelaus's Theorem, the ratio $\frac{|BX|}{|CX|}$ should really be $\frac{\left|\overrightarrow{BX}\right|}{\left|\overrightarrow{CX}\right|} = -\frac{\left|\overrightarrow{BX}\right|}{\left|\overrightarrow{XC}\right|}$, and treating the others similarly, the 1 changes to -1.

Moreover, Ceva's Theorem extends naturally to the case of two "exterior" cevians (with the caveat that it is possible to now choose the cevians all parallel — see Figure 3).[2]. Similarly, Menelaus's Theorem extends naturally to the case where all three points are outside the triangle. See Figure 4. Therefore, the final versions are as follows.

Theorem 1.3 (Ceva's Theorem, Final Version)**.** *Choose X on the line $\overleftrightarrow{BC}$, but not B or C; choose Y on the line $\overleftrightarrow{AC}$, but not A or C, and choose Z on the line $\overleftrightarrow{AB}$. but not B or C .*

[2]Of course, this means that the cevians are "concurrent at ∞"

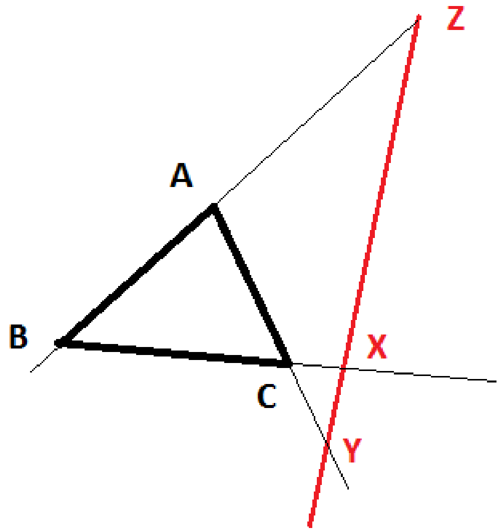

Figure 4: Another case of Menelaus's Theorem

Ceva's Theorem *If the cevians* $\overline{AX}$, $\overline{BY}$, *and* $\overline{CZ}$ *are concurrent or all parallel, then*

$$\frac{\left|\overrightarrow{BX}\right|}{\left|\overrightarrow{XC}\right|} \cdot \frac{\left|\overrightarrow{CY}\right|}{\left|\overrightarrow{YA}\right|} \cdot \frac{\left|\overrightarrow{AZ}\right|}{\left|\overrightarrow{ZB}\right|} = 1. \tag{1.3}$$

Converse *If the points* X, Y, Z *are chosen as above, and if*

$$\frac{\left|\overrightarrow{BX}\right|}{\left|\overrightarrow{XC}\right|} \cdot \frac{\left|\overrightarrow{CY}\right|}{\left|\overrightarrow{YA}\right|} \cdot \frac{\left|\overrightarrow{AZ}\right|}{\left|\overrightarrow{ZB}\right|} = 1,$$

then the cevians $\overline{AX}$, $\overline{BY}$, *and* $\overline{CZ}$ *are concurrent or all parallel.*

Theorem 1.4 (Menelaus's Theorem, Final Version)**.** *Choose* X *on the line* $\overleftrightarrow{BC}$, *but not* B *or* C*; choose* Y *on the line* $\overleftrightarrow{AC}$, *but not* A *or* C, *and choose* Z *on the line* $\overleftrightarrow{AB}$*. but not* B *or* C *.*

Menelaus's Theorem *If* X, Y, *and* Z *are collinear, then*

$$\frac{\left|\overrightarrow{BX}\right|}{\left|\overrightarrow{XC}\right|} \cdot \frac{\left|\overrightarrow{CY}\right|}{\left|\overrightarrow{YA}\right|} \cdot \frac{\left|\overrightarrow{AZ}\right|}{\left|\overrightarrow{ZB}\right|} = -1. \tag{1.4}$$

Converse *If the points X, Y, Z are chosen as above, and if*

$$\frac{\left|\overrightarrow{BX}\right|}{\left|\overrightarrow{XC}\right|} \cdot \frac{\left|\overrightarrow{CY}\right|}{\left|\overrightarrow{YA}\right|} \cdot \frac{\left|\overrightarrow{AZ}\right|}{\left|\overrightarrow{ZB}\right|} = -1,$$

then X, Y, and Z are collinear.

2 Proofs

The above theorems have many, many proofs, especially for the basic versions. See [CG67], [Sil01], [Bog99], [Bog14], [Pam11] among others.

3 Consequences

From Ceva's Theorem, we get the following corollaries.

Corollary 3.1. *The medians of a triangle are concurrent.*

Proof. This follows from the basic Ceva's Theorem. By definition, $|BX| = |XC|$ if $\overline{AX}$ is a median, so $\dfrac{|BX|}{|XC|} = 1$, and similarly for the other ratios. □

Corollary 3.2. *The altitudes of a triangle are concurrent.*

Corollary 3.3. *The (interior) angle bisectors of a triangle are concurrent.*

Corollary 3.4. *Let ΔABC be a triangle, and let X on $\overline{BC}$, Y on $\overline{CA}$, and Z on $\overline{AB}$ be the points of tangency of the circle inscribed in ΔABC. Then $\overline{AX}$, $\overline{BY}$, and $\overline{CZ}$ are concurrent.*

4 For Further Reading

Ceva's theorem and Menelaus's Theorem are actually equivalent; for an elementary proof of their equivalence, see [Sil00].

Ceva's theorem and Menelaus's Theorem have proofs by barycentric coordinates, which is effectively a form of projective geometry; see [Sil01], Chapter 4, for a proof using this approach (and Chapter 9.2 for one of the most accessible expositions of projective geometry I have seen). For other projective-geometry proofs, see [Gre57] and [Ben07].

For a higher-dimensional extension, see [Lan88].

References

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