The Theorems of Ceva and Menelaus

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1 Introduction

In their most basic form, Ceva’s Theorem and Menelaus’s Theorem are simple formulas of triangle geometry. To state them, we require some definitions.

- Three or more line segments in the plane are concurrent if they have a common point of intersection.

1The reverse is true, but complicated. See [Sil00] for more details.
• A cevian of a triangle $\triangle ABC$ is a line segment with one endpoint at one vertex of the triangle (say $A$) and one endpoint on the opposite line (say $\overrightarrow{BC}$), but not passing through the opposite vertices ($B$ or $C$).

We also denote the length of line segment $AB$ to be $|AB|$.

**Theorem 1.1** (Ceva’s Theorem, Basic Version). Choose $X$ on the line segment $\overline{BC}$, $Y$ on the (interior of the) line segment $\overline{AC}$, and $Z$ on the (interior of the) line segment $\overline{AB}$.

Ceva’s Theorem If the cevians $AX$, $BY$, and $CZ$ are concurrent, then

$$\frac{|BX|}{|XC|} \cdot \frac{|CY|}{|YA|} \cdot \frac{|AZ|}{|ZB|} = 1.$$  \hspace{1cm} (1.1)

Converse If the points $X$, $Y$, $Z$ are chosen as above, and if

$$\frac{|BX|}{|XC|} \cdot \frac{|CY|}{|YA|} \cdot \frac{|AZ|}{|ZB|} = 1,$$

then the cevians $\overline{AX}$, $\overline{BY}$, and $\overline{CZ}$ are concurrent.

See Figure 1.

**Theorem 1.2** (Menelaus’s Theorem, Basic Version). Choose $X$ on the line $\overrightarrow{BC}$ but not on the segment $\overline{BC}$, choose $Y$ on the (interior of the) line segment $\overline{AC}$, and $Z$ on the (interior of the) line segment $\overline{AB}$.

Figure 1: The basic case of Ceva’s Theorem
Menelaus’s Theorem If \( X, Y, \) and \( Z \) are collinear, then

\[
\frac{|BX|}{|CX|} \cdot \frac{|CY|}{|AY|} \cdot \frac{|AZ|}{|BZ|} = 1. \tag{1.2}
\]

Converse If the points \( X, Y, \) and \( Z \) are chosen as above, and if

\[
\frac{|BX|}{|CX|} \cdot \frac{|CY|}{|AY|} \cdot \frac{|AZ|}{|BZ|} = 1,
\]

then \( X, Y, \) and \( Z \) are collinear.

See Figure 1.

Menelaus’s Theorem was known to the ancient Greeks, including Menelaus of Alexandria: a proof comes from Menelaus’s *Spherica* ([OR99]). We have no evidence, however, that Ceva’s theorem was discovered formally before Ceva’s publication of *De Lineas Rectis* in 1678 ([OR12]). Nevertheless, the theorems have a certain similarity.

In fact, not to put too fine a point on it, except for the placement of \( X \), the equations (1.1) and (1.2) look alike. What is going on here?

The general idea is that ultimately, Ceva’s and Menelaus’s Theorems are theorems about *signed* lengths ([Bog99], [Sil01]). If, for example, \( B, X, \) and \( C \) are collinear, then

\[
\frac{|BX|}{|XC|}
\]

should be considered positive if \( X \) is between \( B \) and \( C \) (i.e., \( \overrightarrow{BX} \) and \( \overrightarrow{XC} \) are in the same direction) and negative otherwise (i.e, \( \overrightarrow{BX} \) and \( \overrightarrow{XC} \) are in opposite directions). To
Figure 3: Another case of Ceva’s Theorem

make this “signed length” clear, we will use the symbol $\overrightarrow{BX}$, and it will give the number $\lambda$ such that the vectors $\overrightarrow{BX}$, $\overrightarrow{XC}$ satisfy $\overrightarrow{BX} = \lambda \cdot \overrightarrow{XC}$. Then in Menelaus’s Theorem, the ratio $\frac{|BX|}{|CX|}$ should really be $\frac{|BX|}{|CX|} = -\frac{|BX|}{|XC|}$, and treating the others similarly, the 1 changes to $-1$.

Moreover, Ceva’s Theorem extends naturally to the case of two “exterior” cevians (with the caveat that it is possible to now choose the cevians all parallel — see Figure 3).\footnote{Of course, this means that the cevians are “concurrent at $\infty$”}

Similarly, Menelaus’s Theorem extends naturally to the case where all three points are outside the triangle. See Figure 4. Therefore, the final versions are as follows.

**Theorem 1.3** (Ceva’s Theorem, Final Version). Choose $X$ on the line $\overline{BC}$, but not $B$ or $C$; choose $Y$ on the line $\overline{AC}$, but not $A$ or $C$, and choose $Z$ on the line $\overline{AB}$, but not $B$ or $C$.\footnote{Of course, this means that the cevians are “concurrent at $\infty$”}
Ceva’s Theorem If the cevians $AX$, $BY$, and $CZ$ are concurrent or all parallel, then

$$\frac{BX}{XC} \cdot \frac{CY}{YA} \cdot \frac{AZ}{ZB} = 1.$$  \hspace{1cm} (1.3)

Converse If the points $X$, $Y$, $Z$ are chosen as above, and if

$$\frac{BX}{XC} \cdot \frac{CY}{YA} \cdot \frac{AZ}{ZB} = 1,$$

then the cevians $AX$, $BY$, and $CZ$ are concurrent or all parallel.

Theorem 1.4 (Menelaus’s Theorem, Final Version). Choose $X$ on the line $BC$, but not $B$ or $C$; choose $Y$ on the line $AC$, but not $A$ or $C$, and choose $Z$ on the line $AB$. but not $B$ or $C$.

Menelaus’s Theorem If $X$, $Y$, and $Z$ are collinear, then

$$\frac{BX}{XC} \cdot \frac{CY}{YA} \cdot \frac{AZ}{ZB} = -1.$$  \hspace{1cm} (1.4)
Converse If the points $X$, $Y$, $Z$ are chosen as above, and if

$$\frac{|BX|}{XC} \cdot \frac{|CY|}{YA} \cdot \frac{|AZ|}{ZB} = -1,$$

then $X$, $Y$, and $Z$ are collinear.

2 Proofs

The above theorems have many, many proofs, especially for the basic versions. See [CG67], [Sil01], [Bog99], [Bog14], [Pam11] among others.

3 Consequences

From Ceva’s Theorem, we get the following corollaries.

Corollary 3.1. The medians of a triangle are concurrent.

Proof. This follows from the basic Ceva’s Theorem. By definition, $|BX| = |XC|$ if $AX$ is a median, so $\frac{|BX|}{XC} = 1$, and similarly for the other ratios.

Corollary 3.2. The altitudes of a triangle are concurrent.

Corollary 3.3. The (interior) angle bisectors of a triangle are concurrent.

Corollary 3.4. Let $\Delta ABC$ be a triangle, and let $X$ on $BC$, $Y$ on $CA$, and $Z$ on $AB$ be the points of tangency of the circle inscribed in $\Delta ABC$. Then $AX$, $BY$, and $CZ$ are concurrent.

4 For Further Reading

Ceva’s theorem and Menelaus’s Theorem are actually equivalent; for an elementary proof of their equivalence, see [Sil00].

Ceva’s theorem and Menelaus’s Theorem have proofs by barycentric coordinates, which is effectively a form of projective geometry; see [Sil01]. Chapter 4, for a proof using this approach (and Chapter 9.2 for one of the most accessible expositions of projective geometry I have seen). For other projective-geometry proofs, see [Gre57] and [Ben07].

For a higher-dimensional extension, see [Lan88].
References


