What is . . . tetration?

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1 Tetration

1.1 Introduction and Knuth's up-arrow notation

For positive integers, the operations of multiplication and exponentiation can be looked at as repeated addition and repeated multiplication respectively. It is possible to further this process and create an operation that corresponds to repeated exponentiation. These operations are called hyper operations. [GD] [GR] [WH]

Hyper operation rank	Name	Operation	Definition
1	Addition	$a+b$	$a+\underbrace{1+\cdots+1}_{b}$
2	Multiplication	$a\cdot b$	$\underbrace{a+\cdots+a}_{b}$
3	Exponentiation	a^b, $\quad a\uparrow b$	$\underbrace{a\cdot\cdots\cdot a}_{b}$
4	Tetration	${}^{b}a$, $\quad a\uparrow\uparrow b$	$\underbrace{a\uparrow(a\uparrow(\cdots(a\uparrow a)\cdots))}_{b\text{ occurrences of }a}$
5	Pentation	${}_{b}a$, $\quad a\uparrow\uparrow\uparrow b$	$\underbrace{a\uparrow\uparrow(a\uparrow\uparrow(\cdots(a\uparrow\uparrow a)\cdots))}_{b\text{ occurrences of }a}$
6	Hexation	a_b, $\quad a\uparrow\uparrow\uparrow\uparrow b$	$\underbrace{a\uparrow\uparrow\uparrow(a\uparrow\uparrow\uparrow(\cdots(a\uparrow\uparrow\uparrow a)\cdots))}_{b\text{ occurrences of }a}$
$\vdots$	$\vdots$	$\vdots$	$\vdots$

The names of "tetration," "pentation," etc. are from Goodstein (1947). The arrows are part of Knuth's up-arrow notation, created by Donald Knuth in 1976 [WK] [KD], which is defined as in the table. The general rule for this notation is that each operator is defined by the one below it by the following equation:

$$a\underbrace{\uparrow\cdots\uparrow}_{n}b=\underbrace{a\underbrace{\uparrow\cdots\uparrow}_{n-1}\left(a\underbrace{\uparrow\cdots\uparrow}_{n-1}\left(\cdots\left(a\underbrace{\uparrow\cdots\uparrow}_{n-1}a\right)\cdots\right)\right)}_{b\text{ occurrences of }a}.$$

The shorthand $\uparrow^n$ is used to represent $\underbrace{\uparrow\cdots\uparrow}_{n}$. In the expression $a\uparrow^n b$, a is called the base, b is called the hyperexponent, and n is called the rank [WH]. Note that by definition, $x\uparrow^n 1=1$ for all $n\in\mathbb{N}$ (also true for multiplication) and $2\uparrow^n 2=2\uparrow^{n-1}2=\cdots=4$ for all $n\in\mathbb{N}$ (also true for addition and multiplication). Also, like exponentiation, tetration is not commutative. An immediate consequence of the definition of tetration is that ${}^{b+1}a=a^{\left({}^{b}a\right)}$ for $b\geq 1$.

Example 1.1. ${}^{3}2=2^{2^{2}}=16\neq 27=3^3={}^{2}3$.

As one might expect from comparing addition, multiplication, and exponentiation, tetration of even small numbers can result in very large numbers. Here are two tables of values formed by hyper operations displayed using notation no "higher" than exponentiation (and braces) to give an idea of their sizes. Graphs of $y=x+2$, $y=x\cdot 2$, $y=x^2$, and $y={}^{2}x$ are given in **Figure 1**.

rank n	Name	$2\uparrow^{n-2} 2$	$3\uparrow^{n-2} 2$	$4\uparrow^{n-2} 2$	$5\uparrow^{n-2} 2$	$10\uparrow^{n-2} 2$
0	Successor	3	3	3	3	3
1	Addition	4	5	6	7	12
2	Multiplication	4	6	8	10	20
3	Exponentiation	4	9	16	25	100
4	Tetration	4	27	64	3125	10^{10}
5	Pentation	$2^2 = 4$	$3^{3^3} \approx 7.63 \cdot 10^{12}$	$\underbrace{4^{\cdot^{\cdot^{\cdot^{4}}}}}_{4} = 4^{4^{256}}$	$\underbrace{5^{\cdot^{\cdot^{\cdot^{5}}}}}_{5} = 5^{5^{5^{3125}}}$	$\underbrace{10^{\cdot^{\cdot^{\cdot^{10}}}}}_{10}$
6	Hexation	$2^2 = 4$	$\underbrace{3^{\cdot^{\cdot^{\cdot^{3}}}}}_{3^{3^3}}$	$\underbrace{4^{\cdot^{\cdot^{\cdot^{4}}}}}_{\underbrace{4^{\cdot^{\cdot^{\cdot^{4}}}}}_{\underbrace{4^{\cdot^{\cdot^{\cdot^{4}}}}}_{4}}}$	$\underbrace{5^{\cdot^{\cdot^{\cdot^{5}}}}}_{\underbrace{5^{\cdot^{\cdot^{\cdot^{5}}}}}_{\underbrace{5^{\cdot^{\cdot^{\cdot^{5}}}}}_{\underbrace{5^{\cdot^{\cdot^{\cdot^{5}}}}}_{5}}}}$	$\underbrace{10^{\cdot^{\cdot^{\cdot^{10}}}}}_{\underbrace{\vdots}_{\underbrace{10^{\cdot^{\cdot^{\cdot^{10}}}}}_{10}}}$, 9 occurrences of $10^{\cdot^{\cdot^{\cdot^{10}}}}$

rank n	Name	$2\uparrow^{n-2} 2$	$2\uparrow^{n-2} 3$	$2\uparrow^{n-2} 4$	$2\uparrow^{n-2} 5$	$2\uparrow^{n-2} 10$
0	Successor	3	4	5	6	11
1	Addition	4	5	6	7	12
2	Multiplication	4	6	8	10	20
3	Exponentiation	4	8	16	32	$\approx 1.27 \cdot 10^{30}$
4	Tetration	$2^2 = 4$	$2^{2^2} = 16$	$\underbrace{2^{\cdot^{\cdot^{\cdot^{2}}}}}_{4} = 65536$	$\underbrace{2^{\cdot^{\cdot^{\cdot^{2}}}}}_{5} = 2^{65536}$	$\underbrace{2^{\cdot^{\cdot^{\cdot^{2}}}}}_{10} = 2^{2^{2^{2^{2^{2^{65536}}}}}}$
5	Pentation	$2^2 = 4$	$65536 = \underbrace{2^{\cdot^{\cdot^{\cdot^{2}}}}}_{4}$	$\underbrace{2^{\cdot^{\cdot^{\cdot^{2}}}}}_{\underbrace{2^{\cdot^{\cdot^{\cdot^{2}}}}}_{4}}$	$\underbrace{2^{\cdot^{\cdot^{\cdot^{2}}}}}_{\underbrace{2^{\cdot^{\cdot^{\cdot^{2}}}}}_{\underbrace{2^{\cdot^{\cdot^{\cdot^{2}}}}}_{4}}}$	$\underbrace{2^{\cdot^{\cdot^{\cdot^{2}}}}}_{\underbrace{\vdots}_{\underbrace{2^{\cdot^{\cdot^{\cdot^{2}}}}}_{4}}}$, 8 occurrences of $2^{\cdot^{\cdot^{\cdot^{2}}}}$
6	Hexation	$2^2 = 4$	$\underbrace{2^{\cdot^{\cdot^{\cdot^{2}}}}}_{\underbrace{2^{\cdot^{\cdot^{\cdot^{2}}}}}_{4}}$	Exercise	* Exercise	Bonus Exercise

Remark 1.2. 2^{10} written using only addition is

$$
\begin{aligned}
\underbrace{2\cdot(2\cdot(\cdots(2\cdot 2)\cdots))}_{10} &= 2\cdot(2\cdot(2\cdot(2\cdot(2\cdot(2\cdot(2\cdot(2\cdot(2\cdot 2)))))))) \\
&= 2\cdot\left(2\cdot\left(2\cdot\left(2\cdot\left(2\cdot\left(2\cdot\left(2\cdot\left(\underbrace{2+\cdots+2}_{4}\right)\right)\right)\right)\right)\right)\right) \\
&= \underbrace{2+\cdots+2}_{\underbrace{\vdots}_{\underbrace{2+\cdots+2}_{4}}}, \text{8 occurrences of } 2+\cdots+2.
\end{aligned}
$$

Many other notations exist that denote both tetration and other hyper operators. One other will be presented below.

1.2 Conway's chained arrow notation

This notation was created by John Conway [WC]. This notation can be used to write numbers that are too large to concisely write using Knuth's up-arrow notation.

Definition 1.3. [WC] A Conway chain is an expression of the form $a_1 \to a_2 \to \cdots \to a_n$, where the a_i's are positive integers. The length of the chain is the number of numbers in the chain. This chain behaves according to four rules. Let C represent a Conway chain.

1. The Conway chain a is the number a.
2. $a \to b := a^b$.
3. $C \to a \to 1 = C \to a$.
4. $C \to a \to (b+1) = C \to (\cdots(C \to (C \to (C) \to b) \to b)\cdots) \to b$, where the expression on the right has a copies of C.

Each of these rules evaluates a chain in terms of operations no "higher" than exponentiation, reduces the length of the chain, or reduces the final element of the chain. It follows that a Conway chain can always be evaluated to result in an integer, even though it may be quite large.

Example 1.4. Conway's notation can be considered as a kind of superset of Knuth's notation, as a chain of length 3 is equivalent to an expression in Knuth's notation.

- $a \to b \to n = a \uparrow^n b$.
- $4 \to 3 \to 2 = 4 \to (4 \to 4 \to 1) \to 1 = 4 \to (4 \to 4) = 4 \to 4^4 = 4^{4^4} \approx 1.34 \cdot 10^{154}$.
- $3 \to 2 \to 2 \to 2 = 3 \to 2 \to (3 \to 2) \to 1 = 3 \to 2 \to (3 \to 2) = 3 \to 2 \to 9 = 3 \uparrow^9 2$.
- $4 \to 3 \to 2 \to 2 = 4 \to 3 \to (4 \to 3) \to 1 = 4 \to 3 \to 4^3 = 4 \uparrow^{4^3} 3$.

As a further illustration, we will express Graham's number using these notations. [MG]

Definition 1.5. Consider a hypercube of dimension n and color each edge of the hypercube one of two colors. Let N be the smallest n such that no matter what the coloring, a graph K_4 consisting of one color and with coplanar vertices is formed. Graham's number is an upper bound for N found by Graham and Rothschild (1971), which is defined as g_{64} where

$$g_1 = 3 \uparrow\uparrow\uparrow\uparrow 3 \qquad \text{and} \qquad g_n = 3\underbrace{\uparrow \cdots \uparrow}_{g_{n-1}}3.$$

This number satisfies

$$3 \to 3 \to 64 \to 2 < g_{64} < 3 \to 3 \to 65 \to 2.$$

1.3 Numbers of the form ${}^n x$

In accordance with the equation $x^2 = a$, $a \in \mathbb{C}$, one can consider the equation ${}^2x = a$. Much like addition, multiplication, and exponentiation, tetration has an inverse [B], and following exponentiation, it has two inverses corresponding to the root and logarithm of exponentiation [WT].

Definition 1.6. We say that a is an n'th super-root of b if ${}^n a = b$.

We denote a by $\sqrt[n]{b}_s$ or $\sqrt{b}_s$ if $n = 2$.

Definition 1.7. The super-logarithm function slog is defined such that $\mathrm{slog}_a b = n$ if ${}^n a = b$.

While $x = \sqrt{a}_s$ satisfies ${}^2x = a$, one can solve this equation in terms of more familiar functions, specifically ones that were not designed to solve precisely this equation.

Definition 1.8. [CGHJK] The Lambert W function is the (multivalued) inverse function of $f(z) = ze^z$ ($z \in \mathbb{C}$).

A value of the function at z is denoted by $W(z)$ (that is, $W(z)$ satisfies $z = W(z)\, e^{W(z)}$). Various branches of W, each a normal function, are denoted by subscripts such as W_0 and W_{-1} [CGHJK]. This function has numerous applications in mathematics, and one of them is to solve equations of the form $x^x = a$. Any solution satisfies the following equations:

$$\begin{aligned} x \log x &= \log a \\ (\log x)\, e^{\log x} &= \log a \\ \log x &= W(\log a) \\ x &= e^{W(\log a)}. \end{aligned}$$

One question about numbers of the form ${}^n x$ is whether or not there exist irrational numbers a such that ${}^n a$ is rational. Note that $\left(\sqrt[n]{2}\right)^n = 2$ so the statement is true for numbers of the form a^n.

Theorem 1.9. *[AT] Every rational number $r \in \left(\left(\frac{1}{e}\right)^{\frac{1}{e}}, \infty\right)$ either belongs to the set $\left\{1^1, 2^2, 3^3, \ldots\right\}$ or is of the form a^a for an irrational a.*

Proof. Let $f(x) = x^x$. Since f is increasing on $\left(\frac{1}{e}, \infty\right)$ and $\lim_{x\to\infty} f(x) = \infty$, for each $r \in \left(\left(\frac{1}{e}\right)^{\frac{1}{e}}, \infty\right)$ there is a unique $a \in \mathbb{R}$ such that $r = a^a$. Suppose that a is rational, and write $a = \frac{b}{c}$ and $r = \frac{s}{t}$ in lowest terms. Then

$$\begin{aligned} \left(\frac{b}{c}\right)^{\frac{b}{c}} &= \frac{s}{t} \\ \frac{b^{\frac{b}{c}}}{c^{\frac{b}{c}}} &= \frac{s}{t} \\ b^{\frac{b}{c}} t &= s c^{\frac{b}{c}} \\ b^b t^c &= s^c c^b. \end{aligned}$$

Since $(b, c) = (s, t) = 1$, a prime p divides t^c iff p divides c^b, and then p cannot divide b^b or s^c. Similarly, a prime q divides b^b iff q divides s^c, and then q cannot divide t^c or c^b. Then $b^b t^c = s^c c^b = p_1^{e_1} \cdots p_m^{e_m} q_1^{f_1} \cdots q_n^{f_n}$ where $p_1, \ldots, p_m$ divide only t^c and c^b and $q_1, \ldots, q_n$ divide only b^b and s^c, so $t^c = c^b$. Suppose that $c > 1$. There exists a prime p, positive integers i, j, k, and l where k and l are relatively prime to q, so that $t = p^i k$ and $c = p^j l$. Then since $\left(p^i k\right)^c = \left(p^j l\right)^b$, we have

$$jb = ic = ip^j l,$$

which implies that p^j divides jb. But since p divides c and $(b, c) = 1$, p^j actually divides j, so $p^j \le j$. But we know that

$$p^j \ge 2^j > j \qquad \text{for } j \in \mathbb{N},$$

which is a contradiction. Thus $c = 1$. ∎

The method of proof for the ${}^3 x$ case uses an application of the Gelfond-Schneider Theorem [AT].

Theorem 1.10. *(Gelfond-Schneider) If $a \ne 0, 1$ is an algebraic number and b is an irrational algebraic number, then a^b is transcendental.*

Corollary 1.11. *[AT] Every rational number $r \in (0, \infty)$ either belongs to the set $\left\{1^{1^1}, 2^{2^2}, 3^{3^3}, \ldots\right\}$ or is of the form a^{a^a} for an irrational a.*

Proof. Let $f(x) = x^{x^x}$. First we note that

$$\lim_{x\to 0^+} f(x) = \lim_{x\to 0^+} x^{x^x} = \left(\lim_{x\to 0^+} x\right)^{\lim_{x\to 0^+} x^x} = 0^1 = 0$$

and $f'(x) = x^{x^x} x^{x-1}\left(1 + x(\log x)(1+\log x)\right)$. x^{x^x} and x^{x-1} are positive on $(0,\infty)$,

$$\begin{cases} 1 + x(\log x)(1+\log x) \geq 1 & 0 < x \leq \frac{1}{e} \text{ or } 1 \leq x \\ 1 + x(\log x)(1+\log x) > 1 + (\log x)(1+\log x) = \left(\log x + \frac{1}{2}\right)^2 + \frac{3}{4} > 0 & \frac{1}{e} < x < 1 \end{cases},$$

and $f(x) \to \infty$. So f is a bijection from $(0,\infty)$ to $(0,\infty)$. Now let $r \in \mathbb{Q}^+$ and a be the unique number such that ${}^3a = r$. We can assume that a is rational, so we write $a = \frac{b}{c}$ in lowest terms and define $\alpha = \frac{b}{c}$ and $\beta = \left(\frac{b}{c}\right)^{\frac{b}{c}}$. Suppose that $m > 1$. Then β is a root of the polynomial $p(x) = c^b x^c - b^b$. By Theorem 1.9, β is irrational. So by the Gelfond-Schneider Theorem, $r = {}^3a = \alpha^\beta$ is transcendental, thus irrational. So a must be an integer if it is rational. $\square$

Note that in contrast, $\left(\frac{1}{2}\right)^n = \frac{1}{2^n}$ is rational.

Remark 1.12. The proof appears to still hold if "rational number" in the statement of the corollary is replaced with "algebraic."

According to [AT] the question for ${}^na$, $n \geq 4$, is open. However, they conjecture a result similar to the cases $n = 2$ and $n = 3$.

Conjecture 1.13. *For $n \geq 4$, every rational number $r \in (0,\infty)$ either belongs to the set $\{{}^n1, {}^n2, {}^n3, \ldots\}$ or is of the form ${}^na$ for an irrational a.*

1.4 Extensions

Tetration can be extended to bases that are not positive integers. Following the pattern $\lim_{x\to 0} {}^nx = \begin{cases} 1 & n \text{ even} \\ 0 & n \text{ odd} \end{cases}$ (evidence for this assertion is in the next section), one can define ${}^n0 = 1$. For the complex number i, we have

$$\begin{aligned} i^{a+bi} &= e^{(a+bi)\log i} \\ &= e^{\frac{1}{2}\pi i(a+bi)} \\ &= e^{-\frac{b}{2}\pi}\left(\cos\frac{a}{2}\pi + i\sin\frac{a}{2}\pi\right), \end{aligned}$$

so if ${}^ni = a + bi$ then ${}^{n+1}i = e^{-\frac{b}{2}\pi}\cos\frac{a}{2}\pi + ie^{-\frac{b}{2}\pi}\sin\frac{a}{2}\pi$. This method works similarly for a general complex number [WT]. Tetration can also be extended, in a limited way, to hyperexponents that are not positive integers. Let $x > 0$ and $a, b \in \mathbb{N}$. Since ${}^{n+1}x = x^{({}^nx)}$ for $n \geq 1$, we have $\log_x {}^{n+1}x = {}^nx$ for $n \geq 1$. Then one can extend this rule to $n = 0$ to define ${}^0x$ as $\log_x {}^1x = \log_x x = 1$. Continuing this process gives ${}^{-1}x := \log_x {}^0x = \log_x 1 = 0$ and ${}^{-2}x := \log_x {}^{-1}x = \log_x 0$, which is undefined. Thus ${}^nx$ cannot be extended in this way to integers -2 or smaller [B]. When x is multiplied by or raised to the power of a rational number, the rules of $x \cdot \frac{a}{b} = \frac{ax}{b}$ and $x^{\frac{a}{b}} = \sqrt[b]{x^a}$, which involve inverse functions, hold. So for tetration, one may attempt to define ${}^{\frac{a}{b}}x$ to be $\sqrt[b]{{}^ax}_s$ [B]. In the rules involving multiplication and exponentiation, the result does not change if a and b are multiplied by the same natural number, but that property does not hold in the proposed definition for tetration. Let $y = {}^{\frac{2}{3}}x$ and $\bar{y} = {}^{\frac{4}{6}}x$, then

$$\begin{aligned} y &= \sqrt[3]{{}^2x}_s \\ {}^3y &= {}^2x \end{aligned}$$

and

$$\begin{aligned} \bar{y} &= \sqrt[6]{{}^4x}_s \\ {}^6\bar{y} &= {}^4x \end{aligned}$$

but y and $\bar{y}$ are not necessarily equal, see **Figure 2**. Since the definition of $x \uparrow\uparrow\uparrow n$ contains $x \uparrow\uparrow x = {}^x x$, the definition of pentation for noninteger x if $n \geq 2$ requires that ${}^x x$ be well-defined.

1.5 The infinite power tower $h(x) = x^{x^{x^{\cdot^{\cdot^{\cdot}}}}}$

Proposition 1.14. *Let f be continuous and suppose that the sequence (a_n) defined by*

$$a_1 = x, \qquad a_2 = f(x), \qquad a_3 = f(f(x)), \qquad a_4 = f(f(f(x))), \ldots$$

converges to l. Then $f(l) = l$.

For $x > 0$, consider the sequence (b_n) where $b_1 = x$, $b_2 = x^x$, $b_3 = x^{x^x}$, and so on. The symbol

$$x^{x^{x^{\cdot^{\cdot^{\cdot}}}}}$$

refers to the limit of (b_n) if it exists [SM] [KA]. Let b be this limit, then by the proposition, $x^b = b$. Note that this equation implies $x = b^{\frac{1}{b}}$, so $h(x) = x^{x^{x^{\cdot^{\cdot^{\cdot}}}}}$ is the inverse function of $g(x) := x^{\frac{1}{x}}$ whenever both are well-defined and on a domain where g is injective.

Theorem 1.15. *The function $h(x) = x^{x^{x^{\cdot^{\cdot^{\cdot}}}}}$ $(x > 0)$ converges for $e^{-e} \leq x \leq e^{\frac{1}{e}}$ and diverges elsewhere.*

Proof. (Overview of the proof, [KA]) The sequence (b_n) follows one of four cases depending on the value of x.

1. If $1 \leq x$, then $b_1 \leq b_2 \leq b_3 \leq \cdots$.
   (a) If $1 \leq x \leq e^{\frac{1}{e}}$, then (b_n) is also bounded above by e, thus it converges.
   (b) If $e^{\frac{1}{e}} < x$, then note that if b exists then $x^b = b$ implies that $x = b^{\frac{1}{b}}$, and the function $f(y) = y^{\frac{1}{y}}$ has a maximum at $e^{\frac{1}{e}}$. Thus (b_n) cannot converge.
2. If $0 < x < 1$, then $b_1 < b_3 < b_5 < \cdots$ and $b_2 > b_4 > b_6 > \cdots$. Since $0 < x^y < 1$ for $0 < x < 1$ and $0 < y < 1$, both these subsequences are bounded and thus converge.
   (a) If $e^{-e} \leq x < 1$, then (b_{2n-1}) and (b_{2n}) converge to the same value.
   (b) If $0 < x < e^{-e}$, then (b_{2n-1}) and (b_{2n}) converge to different values.

□

A graph of $f(x) = b_n(x)$, $n = 1, \ldots, 16$, is given in **Figure 3**. The above domain of convergence corresponds to the domain $e^{-1} \leq x \leq e$ of the inverse function g with range $e^{-e} \leq g(x) \leq e^{\frac{1}{e}}$. Convergence also holds for certain complex numbers, once h is extended to $\mathbb{C}$. Since $z^w = e^{w \log z}$, one needs to choose a branch of $\log z$ to use; Thron uses the principal branch [T].

Theorem 1.16. *[T] $h(z)$ converges if $|\log z| \leq \frac{1}{e}$. For such z, $|\log h(z)| \leq 1$.*

(In fact, Thron proves a more general result, where $w = z_1^{z_2^{z_3^{\cdot^{\cdot^{\cdot}}}}}$ converges if $|\log z_k| \leq \frac{1}{e}$ for all k, and then $|\log w| \leq 1$.) Shell provides a number of regions which $h(z)$ converges within, and one of them is formulated below.

Theorem 1.17. *[SD] $h(z)$ converges if $z = e^{\xi e^{-\xi}}$ for some $|\xi| \leq \log 2$.*

While larger than Thron's region, Shell's region does not completely contain Thron's region. In the other direction, Carlsson provides a region outside of which $h(z)$ does not converge [KA].

Theorem 1.18. *[KA] If $h(z)$ converges, then $z = e^{\xi e^{-\xi}}$ for some $|\xi| \leq 1$.*

Figure 4 illustrates these regions. Observe that the region of Carlsson is larger than the union of Thron's and Shell's regions, and this statement is still true if the rest of Shell's regions are included [KA]. An open question is to find the subset of $\mathbb{C}$ such that $h(z)$ converges inside it and diverges outside it. The Lambert W function can also write $h(z) = z^{z^{z^{\cdot^{\cdot^{\cdot}}}}}$ in a closed form expression [CGHJK]. For values of z such that $h(z) = z^{z^{z^{\cdot^{\cdot^{\cdot}}}}}$ converges, the equation $h(z) = z^{h(z)}$ holds and it can be solved in terms of the W function to get

$$h(z) = \frac{W(-\log z)}{-\log z}.$$

1.6 The $z = x^x$ spindle

In this section we will let $\log x$ be the multi-valued complex logarithm, and $\operatorname{Log} x$ be the single-valued principal branch of $\log x$. Recall the definition of x^x in terms of logarithms, $x^x = e^{x \log x}$. While x is restricted to be real, $\log x$ and $z = x^x$ are allowed to be complex, so the complex logarithm is involved. Let $x > 0$. Suppose that $y = \operatorname{Log} x$, then $e^{y+2i\pi n} = e^y = x$ for all $n \in \mathbb{N}$, so $\{\operatorname{Log}|x| + i\pi n\}_{n \text{ even}}$ is the set of values of the complex logarithm of x. Let $x' < 0$. For $y' = \operatorname{Log}|x'|$, we similarly have $\{\operatorname{Log}|x'| + i\pi n\}_{n \text{ odd}}$ as the set of values of the complex logarithm of x'. The resulting family of functions representing x^x is

$$t_n(x) := e^{x(\operatorname{Log}|x| + i\pi n)},$$

where t_n has a domain of $(0, \infty)$ if n is even and a domain of $(-\infty, 0)$ if n is odd. Each t_n is called a thread and the graph of all threads is called the x^x spindle [MM]. A graph of $t_n(x)$ for $n = 0, \ldots, 10$ is given in **Figure 5.1**. In **Figure 5.2** only the $n = 3$ and $n = 4$ threads are displayed. Let $z = u + vi$. If we restrict ourselves to real values of x^x (the xu-plane), then each thread t_n $(n \neq 0)$ takes real values (intersects the plane) whenever $2\pi n x$ is a multiple of 2π, that is, for $x = \frac{k}{n}$. (t_0 takes entirely real values for $0 < x < \infty$.) Indeed, for $p, q \in \mathbb{N}$ and q odd,

$$\left(-\frac{p}{q}\right)^{-\frac{p}{q}} = \left(-\sqrt[q]{\frac{p}{q}}\right)^{-p} = \begin{cases} \left(\frac{p}{q}\right)^{-\frac{p}{q}} & p \text{ even} \\ -\left(\frac{p}{q}\right)^{-\frac{p}{q}} & p \text{ odd} \end{cases},$$

so x^x is defined for multiples of $\frac{1}{q}$ [MM]. Now consider the set of z for a given x. For a fixed $x \in \mathbb{Q}$, write $x = \frac{p}{q}$ in lowest terms. Then $t_n(x) = e^{\frac{p}{q}\operatorname{Log}\left|\frac{p}{q}\right| + in\pi\frac{p}{q}}$ takes one of $2q$ distinct values evenly spaced around the circle $|z| = e^{\frac{p}{q}\operatorname{Log}\left|\frac{p}{q}\right|} = |x|^x$ corresponding to $n = 0, \ldots, 2q-1$. But since n is always even or always odd depending on the sign of x, only q distinct values evenly spaced actually occur for any given x. For a fixed $x \notin \mathbb{Q}$, distinct m and n result in $t_m(x) \neq t_n(x)$ on the circle. Let $\varepsilon > 0$, then by the pigeonhole principle, given enough distinct values of n, there will exist two threads $t_{n'}$ and $t_{n''}$ that satisfy the condition that the distance between $t_{n'}(x)$ and $t_{n''}(x)$ along the circle is less than ε. Then for $k \in \mathbb{N}$, $\left\{t_{n'+(n''-n')k}(x)\right\}$ is a set of points that wrap around the entire circle such that any two consecutive elements have distance less than ε [MM].

Remark 1.19. If one uses this approach to the (complex) square root function represented by $z = \sqrt{x} = e^{\frac{1}{2}(\operatorname{Log}|x| + i\pi n)}$, where n is even for $x > 0$ and odd for $x < 0$, then one gets only two distinct branches for each half ($\{x : x > 0\}$ and $\{x : x < 0\}$) as expected.

2 References

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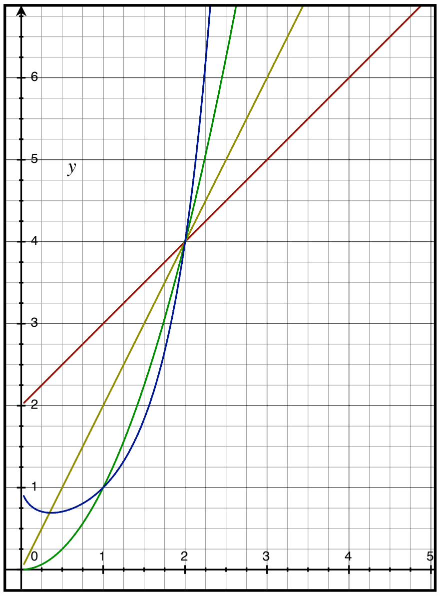
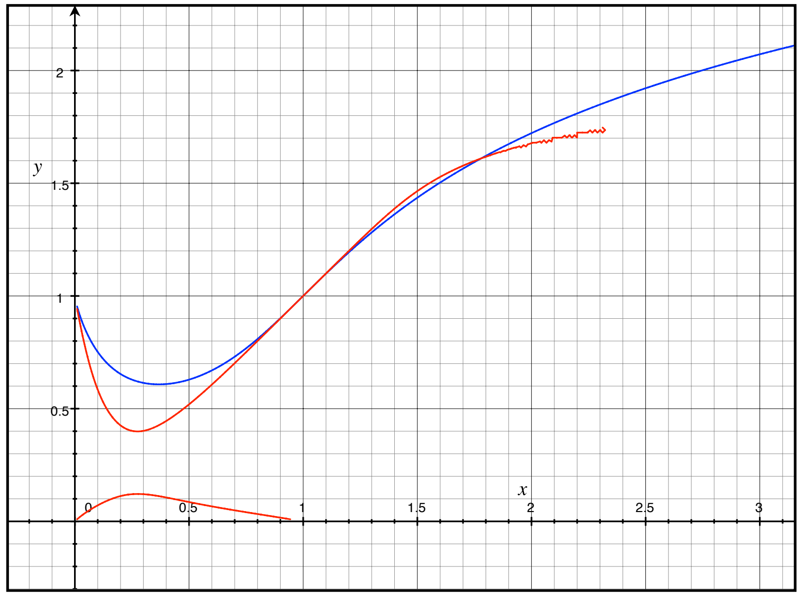
3 List of Figures

Figure 1



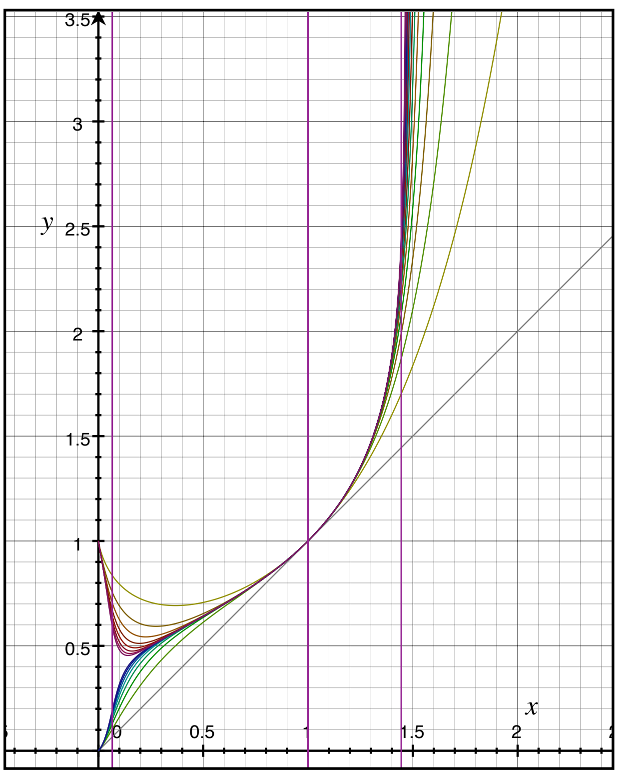
$y = x + 2$, $y = x \cdot 2$, $y = x^2$, and $y = {}^2x$.

Figure 2



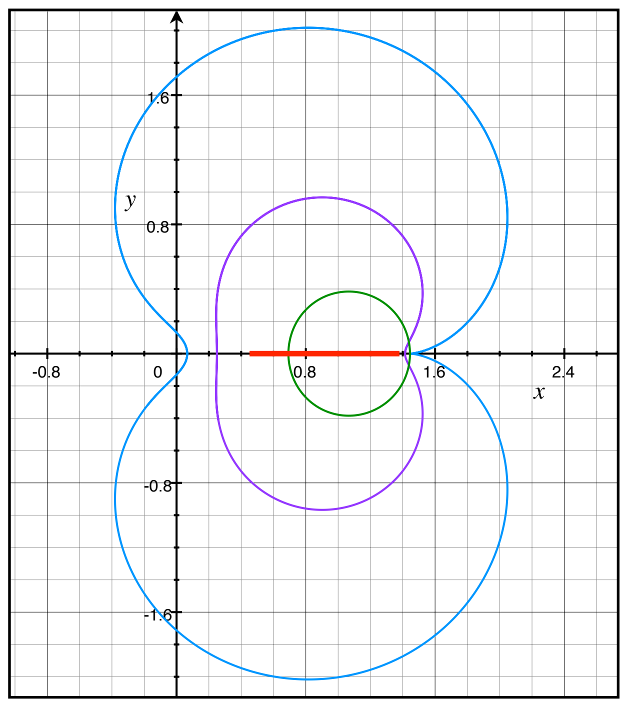
The blue curve is ${}^3y = {}^2x$ and the red curve is ${}^6y = {}^4x$.

Figure 3



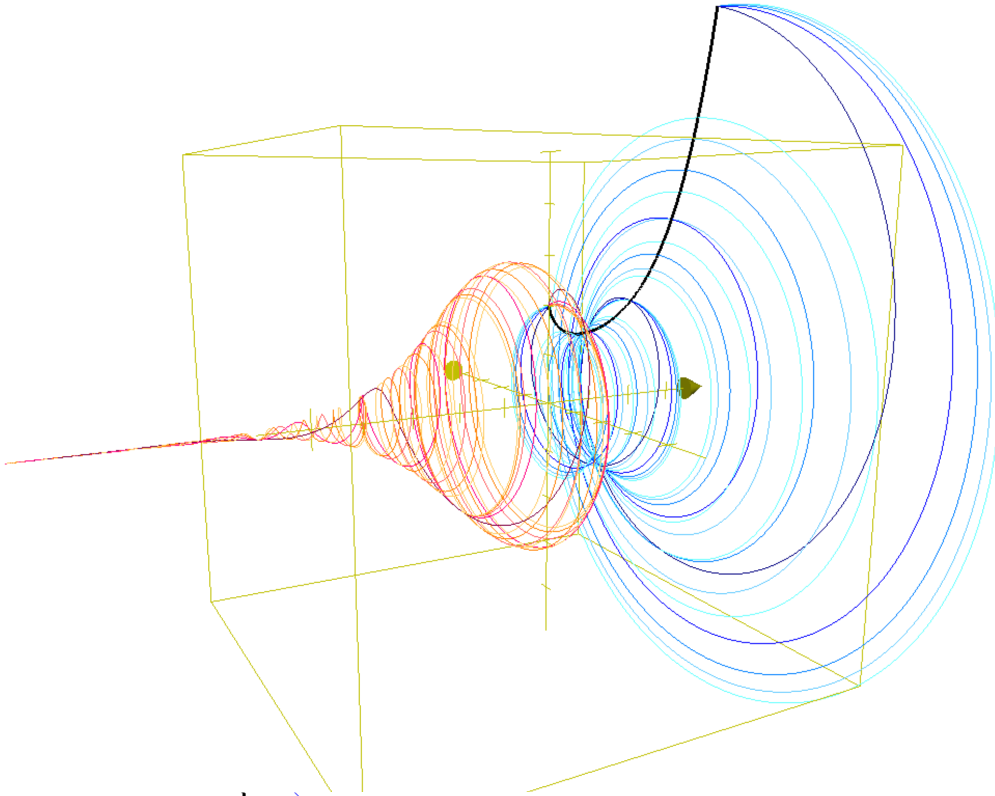
The vertical lines are, from left to right, $x = e^{-e}$, $x = 1$, and $x = e^{1/e}$. Yellow to red curves are $y = b_{2n}$ and green to blue curves are $y = b_{2n-1}$.

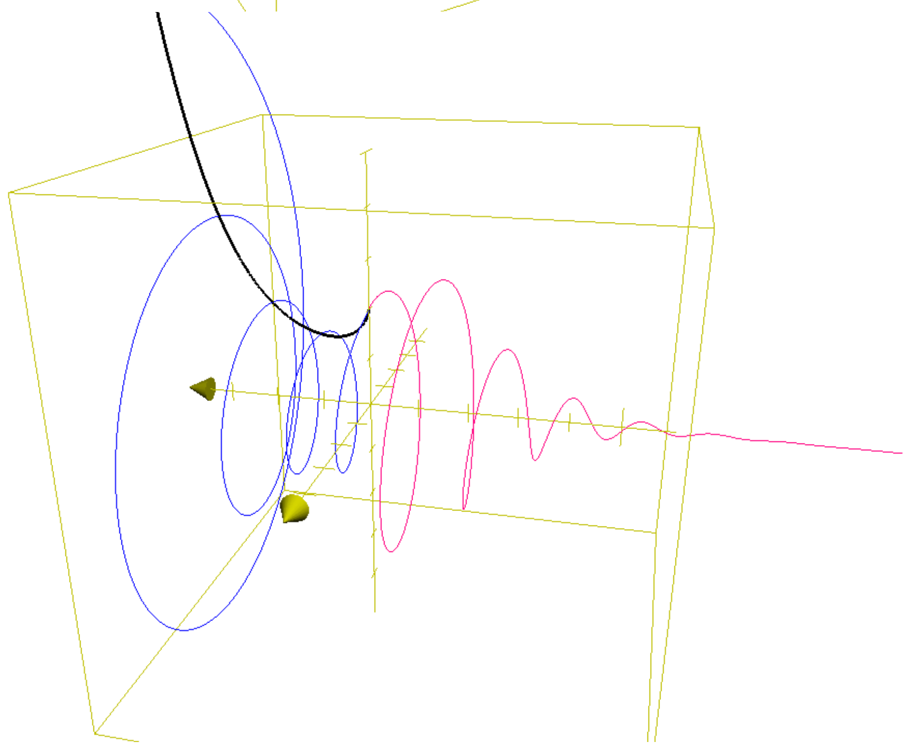
Figure 4



The red line is the convergence interval for real z, the green curve is the boundary of Thron's region, the purple curve is the boundary of the Shell's region mentioned in the handout, and the blue curve is the boundary of Carlsson's region.

Figures 5.1 (top) **and 5.2** (bottom)





The graph of $z = e^{x\cdot\log|x| + in\pi x}$. x *is along the axis in the same direction as the spindle, and* $z = u + vi$ *is the plane perpendicular to that axis, where the vertical axis represents* u*. The frame limits are* $[-3, 2] \times [-2.5, 2.5]^2$*. The thick black curve is* $n = 0$*. [Top] The blue curves are* $n = 2, 4, 6, 8, 10$*, and the reddish curves are* $n = 1, 3, 5, 7, 9$*. [Bottom] The blue curve is* $n = 4$ *and the red curve is* $n = 3$*.*