What is q-Calculus?

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Abstract

In this talk, I will present a q-analog of the classical derivative from calculus. From there, I will prove q-analogs of the binomial theorem and Taylor's theorem. If time permits, I will show some applications of the q-calculus in number theory and physics.

1 q-Derivative

In classical calculus we define the derivative of a function f to be

$$\frac{df}{dx} = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

Equivalently, we could define the derivative to be

$$\frac{df}{dx} = \lim_{q \to 1} \frac{f(qx) - f(x)}{qx - x}$$

Now one might wonder, what happens when we don't take the limit, and just use the expression inside the limit as the definition of our derivative. This leads us into the exciting world of quantum calculus, also known as q-calculus.

Definition 1.1. $q-differential$: A $q-differential$ of a function f is defined to be $d_q f = f(qx) - f(x)$.

From here, we can define our $q - derivative$ to be

$$\frac{d_q f}{d_q x} = \frac{f(qx) - f(x)}{qx - x}$$

Now for an example,

$$\frac{d_q}{d_q x}(x^n) = \frac{q^n x^n - x^n}{qx - x} = \frac{q^n - 1}{q - 1} x^{n-1}$$

Comparing this to the classical result, $\frac{d}{dx}(x^n) = nx^{n-1}$, we can define a q-analog of a number to be $[n] = \frac{q^n - 1}{q-1}$. Comparing $\frac{d^n}{dx^n}(x^n) = n!$ and $\frac{d^k}{dx^k}(x^n) = \binom{n}{k} k! x^{n-k}$ to their q-analogs motivates the following two definitions.

Definition 1.2. $q-factorial$: A $q-factorial$ of a natural number n is defined to be $[n]! = [n] \times [n-1] \times ... \times [1]$ if $n > 0$ and 1 if $n = 0$.

Definition 1.3. $q-factorial$: For natural numbers n and k, the q-factorial is defined to be $\begin{bmatrix} n \\ k \end{bmatrix} = \frac{[n]!}{[k]![n-k]!}$.

2 Taylor's Theorem

Now that we have a q-derivative, it is reasonable to wonder if there are theorems analogous to those from classical calculus. In fact, there is a q-analog of Taylor's theorem, but before proving it we are going to restate the classical Taylor theorem in a slightly weaker form.

Theorem 2.1 (Taylor's theorem)**.** Let $P_0, P_1, ..., P_N$ be polynomials and a be a number such that:

(1) $P_0(a) = 1$ and $P_n(a) = 0$ for $n \geq 1$.
(2) $deg(P_n) = n$.
(3) $\frac{d}{dx} P_n(x) = P_{n-1}(x)$ for $n \geq 1$
Then any polynomial f of degree n can be written in the form

$$f(x) = \sum_{n=0}^{N} \frac{d^n f}{dx^n}\Big|_{x=a} P_n(x)$$

Proof. Let V be the vector space of polynomials of degree $\leq N$. The polynomials $P_0, .., P_N$ are linearly independent by (2) since their degrees are strictly increasing. So they form a basis of V, and we have $f = \sum_{n=1}^{N} c_n P_n$. Now to solve for c_k, we will differentiate k times and evaluate at a.

$$\frac{d^k f}{dx^k}\Big|_{x=a} = \sum_{n=1}^{N} c_n (\frac{d^k}{dx^k} P_n)\Big|_{x=a} = \sum_{n=k}^{N} c_n P_{n-k}(a) = c_k$$

□

Now that we have Taylor's theorem in a different form, we can state the q-analog of Taylor's theorem.

Theorem 2.2 (q-Taylor's theorem)**.** Let $P_0, P_1, ..., P_N$ be polynomials and a be a number such that:

(1) $P_0(a) = 1$ and $P_n(a) = 0$ for $n \geq 1$.
(2) $deg(P_n) = n$.
(3) $\frac{d_q}{d_q x} P_n(x) = P_{n-1}(x)$ for $n \geq 1$
Then any polynomial f of degree n can be written in the form

$$f(x) = \sum_{n=0}^{N} \frac{d_q^n f}{d_q x^n}\Big|_{x=a} P_n(x)$$

Proof. The proof is identical to the proof of Theorem 2.1. □

Although we now have the q-analog of Taylor's theorem, we still need to find the polynomials P_n. Naively one would expect $P_n(x) = \frac{(x-a)^n}{[n]!}$ by analogy with the classical case, but these polynomials fail to satisfy condition (3). From conditions (1) and (2), we must have $P_0 = 1$. In order to satisfy all of the conditions we must then have $P_1 = x - a$. Since $\frac{d_q}{d_q x}x^2 = [2]x$, $P_2 = \frac{x^2}{[2]} - ax - \frac{a^2}{[2]} + a^2 = \frac{(x-a)(x-qa)}{[2]}$ satisfies the necessary conditions. Assuming this trend continues, a reasonable guess would be $P_n = \frac{\prod_{k=0}^{n-1}(x-q^k a)}{[n]!}$. To simplify notation, we will introduce the q-analog of $(x-a)^n$

Definition 2.1. *q*-analog of $(x-a)^n$: The q-analog of $(x-a)^n$ is defined to be $(x-a)_q^n = \prod_{k=0}^{n-1}(x - q^k a)$.

Before proving that these P_n satisfies the conditions in the q-Taylor's theorem, we will need the following lemma.

Lemma 2.3 (q-Product Rule)**.**

$$\frac{d_q}{d_q x}(f(x)g(x)) = f(qx)\frac{d_q g}{d_q x} + g(x)\frac{d_q f}{d_q x}$$

Proof.

$$\begin{aligned} d_q(f(x)g(x)) &= f(qx)g(qx) - f(x)g(x) \\ &= f(qx)g(qx) - f(qx)g(x) + f(qx)g(x) - f(x)g(x) \\ &= f(qx)d_q g(x) + g(x)d_q f(x) \qquad (1) \\ \frac{d_q(f(x)g(x))}{d_q x} &= f(qx)\frac{d_q g}{d_q x} + g(x)\frac{d_q f}{d_q x} \end{aligned}$$

□

I would like to point out that this expression is symmetric in f and g, however the symmetry is hidden in the definition of the q-derivative.

Theorem 2.4. $\frac{d_q}{d_q x}(x-a)_q^n = [n](x-a)_q^{n-1}$

Proof. For $n = 1$ this is clearly true. Suppose this theorem holds for $n-1$, then we have

$$(x-a)_q^n = (x-a)_q^{n-1}(x - q^{n-1}a)$$

Using the product rule,

$$\frac{d_q}{d_q x}(x-a)_q^n = (x-a)_q^{n-1} + (qx - q^{n-1}a)\frac{d_q}{d_q x}(x-a)_q^{n-1}$$

$$\frac{d_q}{d_q x}(x-a)_q^n = (x-a)_q^{n-1} + q(x - q^{n-2}a)[n-1](x-a)_q^{n-2}$$

$$\frac{d_q}{d_q x}(x-a)_q^n = (1+q[n-1])(x-a)_q^{n-1} = [n](x-a)_q^{n-1}$$

So the theorem, holds $\forall n \in \mathbb{N} \bigcup 0$ by induction. $\square$

At this point, I would like to note that the definition of $(x+a)_q^n$ can be extended to negative integers with the following definition, $(x-a)_q^{-n} = \frac{1}{(x-q^{-n}a)_q^n}$.

3 Binomial Theorem and Euler's Identities

Now that we have the q-Taylor's theorem, we can use it to prove two q-analogs of the binomial theorem.

Theorem 3.1 (Gauss' Binomial Theorem)**.**

$$(x+a)_q^n = \sum_{k=0}^{n} q^{\frac{k(k-1)}{2}} \begin{bmatrix} n \\ k \end{bmatrix} a^k x^{n-k}$$

Note: This expression is not symmetric in x and a, because the second value will be multiplied by some powers of q.

Proof.

$$\frac{d_q^k}{d_q x^k}(1+x)_q^n = [n][n-1]...[n-k+1](1+x)_q^{n-k}$$

$$(0+a)_q^{n-k} = (a+0)(qa+0)...(q^{n-k-1}a) = 1 \times q \times q^2 \times ... \times q^{n-k-1} \times a^{n-k} = q^{\frac{(n-k)(n-k-1)}{2}} a^{n-k}$$

Using the q-Taylor's theorem we have

$$(x+a)_q^n = \sum_{k=0}^{n} q^{\frac{(n-k)(n-k-1)}{2}} \begin{bmatrix} n \\ n-k \end{bmatrix} a^{n-k} x^k$$

When we replace k with $n-k$ as the variable of summation,, this simplifies to

$$(x+a)_q^n = \sum_{k=0}^{n} q^{\frac{k(k-1)}{2}} \begin{bmatrix} n \\ k \end{bmatrix} a^k x^{n-k}$$

Since $\begin{bmatrix} n \\ n-k \end{bmatrix} = \begin{bmatrix} n \\ k \end{bmatrix}$. $\square$

Theorem 3.2 (Heine's Binomial Theorem)**.**

$$\frac{1}{(1-x)_q^n} = \sum_{k=0}^{\infty} \frac{[n][n-1]...[n-k+1]}{[k]!} x^k$$

Note: this equality holds as an equivalence of formal power series so issues of convergence are ignored.

Proof. The proof of this theorem is completely analogous to the proof of Gauss's binomial theorem and can be found in full in Kac's textbook *Quantum Calculus*. □

Now that we have these two formulas, one might wonder what happens in the limit where n goes to infinity. In the classical case, both sides go to infinity and we extract no new information, however this is not so in the quantum case.

Theorem 3.3 (Euler's Identities)**.**

$$\prod_{k=0}^{\infty}(1+q^k x) = (1+x)_q^{\infty} = \sum_{k=0}^{\infty} q^{\frac{k(k-1)}{2}} \frac{x^k}{(1-q)(1-q^2)...(1-q^k)}$$

$$\prod_{k=0}^{\infty}\frac{1}{(1-q^k x)} = \frac{1}{(1-x)_q^{\infty}} = \sum_{k=0}^{\infty} \frac{x^k}{(1-q)(1-q^2)...(1-q^k)}$$

for $|q| < 1$.

Proof. We have

$$\lim_{n\to\infty}[n] = \lim_{n\to\infty}(\frac{q^n-1}{q-1}) = \frac{1}{1-q}$$

when $|q| < 1$. And

$$\lim_{n\to\infty}\begin{bmatrix} n \\ k \end{bmatrix} = \lim_{n\to\infty}\frac{(q^n-1)(q^{n-1}-1)...(q^{n-k+1}-1)}{(q-1)(q^2-1)...(q^k-1)} = \frac{1}{(1-q)(1-q^2)...(1-q^k)}$$

Plugging these two results into Gauss' and Heine's binomial theorems gives us the desired identities. □

4 Applications

The two Euler identites derived using the q-calculus have many applications in number theory. A few of them will be listed here and the proofs can be found in Victor Kac's textbook *Quantum Calculus*.

Definition 4.1. The classical partition function $p(n)$ is defined to be the number of ways that a positive integer n can be written as a sum of positive integers, 1 if $n = 0$ and 0 if $n < 0$.

Theorem 4.1. For any positive integer n we have

$$p(n) = p(n-e_1)+p(n-e_{-1})-p(n-e_2)-p(n-e_{-2})+p(n-e_3)+p(n-e_3)+...$$

where $e_n = \frac{3n^2-n}{2}$ are the pentagonal numbers.

It is also interesting to note that the same recursive relationship holds for the number of positive divisors of n.

q-Calculus can also give us some interesting results on partitioning numbers into a sum of squares.

Theorem 4.2. For any positive integer N, the number of ways N can be written as a sum of four squares is equal to $8\times$(sum of the positive divisors of N that are not multiples of 4).

Theorem 4.3. For any positive integer N, the number of ways that N can be written as a sum of two squares is equal to 4(number of positive divisors of N $= 1$ mod 4) - 4(number of positive divisors of $N = 3$ mod 4).

q-Calculus has also seen some applications in physics. In theories of quantum gravity q can be thought of as a parameter related to the exponential of the cosmological constant. So when $q = 1$, there is no gravity and we recover 'classical' quantum mechanics. However, when q is not equal to one, we have a theory of quantum mechanics in a spacetime with constant curvature, ie. a theory where the vacuum has a nonzero energy density. More information about this can be found in the week 183 post of John Baez's blog This Week's Finds in Mathematical Physics.