Duffin-Schaeffer Conjecture

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Introduction

- Diophantine approximation.
  For $\alpha \in Q^c \cap [0,1)$, there are infinitely many integers m, n with $(m,n) = 1$ such that

$$\left|\alpha - \frac{m}{n}\right| < \frac{1}{n^2}$$

- Let $\psi : N \to [0, \infty)$

$$A_\infty = \{\alpha \in [0,1) : \left|\alpha - \frac{m}{n}\right| < \frac{\psi(n)}{n}, (m,n) = 1 \ i.o.\}$$

Question : What is the Lebesque measure of A_∞?

Lebesque Measure μ on $[0, 1)$

- $\mu(I) = b - a$, for $I = [a, b]$, (a, b), $[a, b)$, or $(a, b]$, so $\mu[0, 1) = 1$
- For $A = \cup_k I_k$,
  If it is disjoint, $\mu(A) = \sum_k \mu(I_k)$
  In general, $\mu(A) \leq \sum_k \mu(I_k)$
- A is of measure zero, $\mu(A) = 0$
  if for any ϵ, there is a finite or countable family of intervals I_k such that $A \subset \cup I_k$ and $\sum \mu(I_k) \leq \epsilon$.
- $\mu(A) = 1$ if $\mu(A^c) = 0$.
- A property holds for a.e. x if the set of x it fails to hold the property is a set of measure zero.

Lebesque Measure μ on $[0, 1)$

- Example
  Countable set $A = \{r_k\}$ is of measure zero.
  Consider $I_k = (r_k - \frac{\epsilon}{2^{k+1}}, r_k + \frac{\epsilon}{2^{k+1}})$ and $\mu(I_k) = \frac{\epsilon}{2^k}$
- from Diophantine approximation,
  for almost every α,
  there are infinitely many pairs of m, n with $(m, n) = 1$

$$\left|\alpha - \frac{m}{n}\right| < \frac{1}{n^2}$$

Khintchine's theorem(1924)

Let $\psi : N \to [0, \infty)$ and suppose that $n\psi(n)$ non-increasing.
If $\sum \psi(n) = \infty$, then $\mu(A_\infty) = 1$,i.e, for a.e. α,
there are infinitely many pairs of m, n with $(m, n) = 1$

$$\left|\alpha - \frac{m}{n}\right| < \frac{\psi(n)}{n}$$

However, if $\sum \psi(n) < \infty$, then $\mu(A_\infty) = 0$,i.e, for a.e. α,
there are only finitely many pairs of m, n with $(m, n) = 1$

$$\left|\alpha - \frac{m}{n}\right| < \frac{\psi(n)}{n}$$

Examples of Khintchine's theorem

- $\psi(n) = \frac{1}{n \log n}$, Then $\mu(A_\infty) = 1$
  i.e, for a.e. α,
  there are infinitely many pairs of m, n with $(m, n) = 1$

$$\left|\alpha - \frac{m}{n}\right| < \frac{1}{n^2 \log n}$$

- $\psi(n) = \frac{1}{n(\log n)^{1+\epsilon}}$, Then $\mu(A_\infty) = 0$
  i.e, for a.e. α,
  there are only finitely many pairs of m, n with $(m, n) = 1$

$$\left|\alpha - \frac{m}{n}\right| < \frac{1}{n^2 (\log n)^{1+\epsilon}}$$

A_∞

- Note that

$$A_\infty = \{\alpha : \alpha \in A_n \ i.o\}$$

$$A_n = \{\alpha : \exists m \text{ s.t. } \left|\alpha - \frac{m}{n}\right| < \frac{\psi(n)}{n} \text{ with}(m,n) = 1\}$$
$$= \cup_{\substack{1 \le m \le n \\ (m,n)=1}} I(m,n)$$

where $I(m,n) = \left(\frac{m}{n} - \frac{\psi(n)}{n}, \frac{m}{n} + \frac{\psi(n)}{n}\right)$
So

$$\mu(A_n) \le \sum_m \mu(I_{m,n}) = 2\frac{\psi(n)}{n}\phi(n)$$

and equailty holds for $\psi(n) \le \frac{1}{2}$,

Probability

- $\mu(A)$: the probability of $x \in A$, picking $x \in [0,1)$
- Borel-Cantelli lemma
  (a) If $\sum \mu(A_n) < \infty$, then $\mu(A_\infty) = 0$
  (b) If $\sum \mu(A_n) = \infty$ and $\mu(A_i \cap A_j) = \mu(A_i)\mu(A_j)$.
  then $\mu(A_\infty) = 1$.
- Erdős-Rényi theorem
  $\sum \mu(A_n)$ diverges. Then,

$$\mu(A_\infty) \geq \limsup_N \frac{(\sum_1^N \mu(A_n))^2}{\sum_{1 \leq i,j \leq N} \mu(A_i \cap A_j)}$$

- Corollary of BC Lemma
  If $\sum \frac{\psi(n)}{n}\phi(n)$ converges, for almost every α,
  there are only finitely many m, n with $(m,n) = 1$

$$\left|\alpha - \frac{m}{n}\right| < \frac{\psi(n)}{n}$$

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Duffin-Schaeffer conjecture(1941)

If $\sum \frac{\psi(n)}{n}\phi(n)$ diverges, then $\mu(A_\infty) = 1$,
that is, for almost every α,
there are infinitely many m, n with $(m, n) = 1$

$$\left|\alpha - \frac{m}{n}\right| < \frac{\psi(n)}{n}$$

- Remark
  Duffin-Schaeffer : $\psi(n) \leq \frac{1}{2}$
  so, $\sum \mu(A_n) = \sum \frac{\psi(n)}{n}\phi(n)$
  Pollington-Vaughan : overcome this difficulty.

Cassel's Zero One Law(1950)

Let $\psi : N \to [0, \infty)$.

$$B = \{\alpha \in [0,1) : |\alpha - \frac{m}{n}| < \frac{\psi(n)}{n} \text{ i.o}\}$$

Then, $\mu(B)$ is either 0 or 1.

- Remark
  The proof is based on that $Tx = 2x$ (mod1) is ergodic.
  (T is ergodic if $T^{-1}A = A$, then $\mu(A)$ is 0 or 1.)

Gallagher's Ergodic theorem(1961)

Let $\psi : N \to [0, \infty)$.

$$A_\infty = \{\alpha \in [0,1) : \left|\alpha - \frac{m}{n}\right| < \frac{\psi(n)}{n}, (m,n) = 1 \ i.o.\}$$

Then, $\mu(A_\infty)$ is either 0 or 1.

- Remark

  The proof uses the fact that for all prime p,
  $T(x) = px + \frac{s}{p} \pmod 1$ is ergodic,

- Theorem
  Suppose that
  $\psi(n) \leq \frac{1}{2}$ and $\sum \frac{\psi(n)}{n}\phi(n)$ diverges.
  In addition,

$$\limsup_N \left(\sum_1^N \frac{\psi(n)}{n}\phi(n) \right) \left(\sum_1^N \psi(n) \right)^{-1} \geq c > 0$$

  Then $\mu(A_\infty) = 1$

Lemma

- Let M, N be positive integers and A be a positive number.
  Let k be the number of solutions for
  $0 < |mN - nM| \leq A$ with $1 \leq m \leq M$ and $1 \leq n \leq N$.
  Then $k \leq 2A$
  Proof) Let $M' = \frac{M}{(M,N)}$ and $N' = \frac{N}{(M,N)}$.
  $$0 < |mN' - nM'| \leq \frac{A}{(M,N)}$$
  This solution satisfies $mN' = a \pmod{M'}$ for $1 \leq |a| \leq \frac{A}{(M,N)}$, which has only one solution in $m \pmod{M'}$ for each a.
  So, there are (M, N) number of solutions in m for each value of a. Hence
  $$k \leq 2\frac{A}{(M,N)}(M,N) = 2A$$

Outline of Proof 1

Show that $\mu(A_n \cap A_t) \leq 8\psi(n)\psi(t)$, for $n \neq t$.
k : number of intersections of $I(m,n)$ and $I(s,t)$
L : maximum length of intersection of $I(m,n)$ and $I(s,t)$

- $k \leq 4\max(t\psi(n), n\psi(t))$
  $I(m,n) \cap I(s,t) \neq \emptyset$
  $\Rightarrow 0 < \left|\frac{m}{n} - \frac{s}{t}\right| < \frac{\psi(n)}{n} + \frac{\psi(t)}{t}$
  $\Rightarrow 0 < |mt - sn| \leq t\psi(n) + n\psi(t) \leq 2\max(t\psi(n), n\psi(t))$
- $L \leq$ min of two intervals $= 2\min(\frac{\psi(n)}{n}, \frac{\psi(t)}{t})$

$$\begin{aligned}
\mu(A_n \cap A_t) &\leq k \cdot L \\
&\leq 4\max(t\psi(n), n\psi(t)) \cdot 2\min(\frac{\psi(n)}{n}, \frac{\psi(t)}{t}) \\
&\leq 8\psi(n)\psi(t)
\end{aligned}$$

Outline of Proof 2

apply Erdős-Rényi theorem

- From $\psi(n) \le \frac{1}{2}$, $\mu(A_n) = 2\frac{\psi(n)}{n}\phi(n)$ and $\sum \mu(A_n) = \infty$
- $\sum_{1 \le i,j \le N} \mu(A_i \cap A_j) \le 8 \left(\sum_1^N \psi(n) \right)^2 + \sum_1^N \psi(n)$

$$\mu(A_\infty) \ge \limsup_N \frac{(\sum_1^N \mu(A_n))^2}{\sum_{1 \le i,j \le N} \mu(A_i \cap A_j)}$$
$$\ge \limsup \frac{1}{8} \left(\sum_1^N \frac{\psi(n)}{n} \phi(n) \right)^2 \left(\sum_1^N \psi(n) \right)^{-2} > 0$$

- the Zero One Law $\Rightarrow \mu(A_\infty) = 1$.

Corollary of Duffin-Schaeffer Theorem

Recall that

$$\sum_{p:\text{prime}} \frac{1}{p} = \infty$$

$$\frac{\phi(p)}{p} = 1 - \frac{1}{p} \geq \frac{1}{2}$$

Hence,

$$\sum \frac{1}{p}\frac{\phi(p)}{p} \geq \frac{1}{2}\sum \frac{1}{p}$$

- For almost every α,
  there are infinitely many prime p and integer m such that

$$|\alpha - \frac{m}{p}| < \frac{1}{p^2}$$

- Erdős(1970)

$$\psi(n) = \frac{c}{n} \text{ or } 0$$

- Valler(1978)

$$\psi(n) \leq \frac{c}{n}$$

- Pollington and Vaughan (1990)
  k dimensional version ($k \geq 2$)
  $\sum \left(\psi(n)\frac{\psi(n)}{n}\right)^k$ diverges.
  Then, for a.e. $x = (x_1, \cdots, x_k) \in [0,1)^k$
  $\max(|x_1 n - a_1|, \cdots, |x_k n - a_k|) < \psi(n)$ with $(a_1 \cdots a_k, n) = 1$
  have infintely many solutions.

References

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