Gordon Prize Examination

February 21, 1998

1. Prove that the maximal area of a 1998-gon inscribed in a given circle is achieved for a regular 1998-gon.

2. Prove the inequality $\sin^{1998} x + \cos^{1998} x \geq \frac{1}{2^{998}}$.

3. Let a_n be the integer closest to $\sqrt{n}$. For example, $a_1 = 1$, $a_2 = 1$, $a_3 = 2$. Evaluate

$$\frac{1}{a_1} + \frac{1}{a_2} + \frac{1}{a_3} + \cdots + \frac{1}{a_{1998}}.$$

4. Let a, b, c be complex numbers. Show that if $a + b + c = 0$ and $|a| = |b| = |c|$, then $a^3 = b^3 = c^3$. [Although a picture may help you, your solution requires reasoning, not just a picture.]

5. Let α be a real number. Find the limit:

$$\lim_{n \to \infty} \sin\alpha \ \sin 2\alpha \ \sin 3\alpha \ \cdots \ \sin n\alpha$$

6. Evaluate $\displaystyle\lim_{n\to\infty} \sum_{k=1}^{n} \frac{n}{(k+n)^2}$.

You may take this sheet with you.
Be sure to hand in separately the cover sheet
(with your name, rank, student number, and secret code name).
Put your secret code name at the top of each answer sheet.