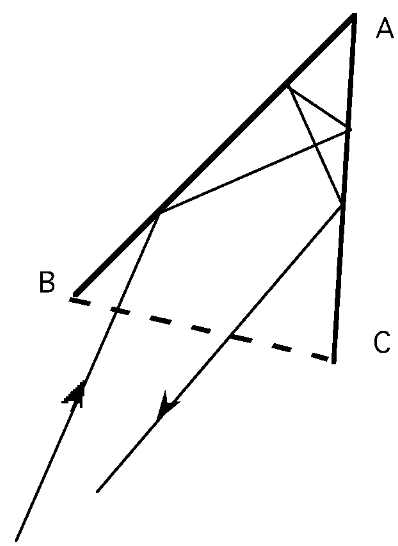
Gordon Prize Examination

February 12, 2000

1. Suppose that each pair $i < j$ from $\{1, 2, \cdots, n\}$ is assigned a direction, either $i \to j$ or $j \to i$. Show that the numbers $1, 2, \cdots, n$ can be listed in some order $a_1, a_2, \cdots, a_n$ so that $a_1 \to a_2 \to a_3 \to \cdots \to a_n$.

2. An angle is formed by two mirrors. If a ray of light enters the angle, and does not hit a corner, then must it leave the angle? More precisely: Suppose the ray crosses side BC into the triangle, and reflects with angle of incidence equal to angle of reflection whenever it hits sides AB and AC. Assume that the ray does not exactly hit any of the points A, B or C. Then must it eventually cross side BC back to the outside?



3. Prove that the number

$$\sum_{n=1}^{\infty} 6^{(2-3n-n^2)/2}$$

is irrational.

4. There are 17 "heavy" points chosen arbitrarily on a circle. At a certain moment they start moving around the circle; they all move with the same constant speed, but some of them may move clockwise and some counterclockwise. When two points meet, they "bounce" by reversing their directions but continuing to move with the same speed. Prove that at a certain moment the points will all return to their starting positions.

5. Let $f(x)$ be a polynomial with integer coefficients. Show that

$$\sum_{n=0}^{\infty} \frac{f(n)}{n!} = ke$$

for some integer k.

6. Let $a_0 > a_1 > a_2 > \cdots > a_n > 0$ be real numbers. Show that the polynomial $p(z) = a_0 + a_1 z + a_2 z^2 + \cdots + a_n z^n$ has no complex zeros z with $|z| \le 1$.

You may take this sheet with you.
Be sure to hand in separately the cover sheet
(with your name, rank, student number, and secret code name).
Put your secret code name at the top of each answer sheet.