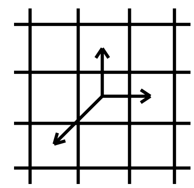
Gordon Prize Examination

February 24, 2001

1. Prove that there is a partition of $\{1, 2, \ldots, 2001\}$ into 667 subsets such that each subset contains exactly 3 elements and, for each subset, the sum of its 3 members is 3003.

2. A *dolphin* is a chess piece that moves either one step forward, or one step to the right, or one step diagonally backwards to the left: Can a dolphin placed in the lower lefthand corner square of an 8×8 chessboard run through all of the chessboard visiting each square exactly once?



3. Prove that there exists a circle in the Cartesian plane $\mathbb{R}^2$ whose interior contains exactly 2001 lattice points (that is, points $(a, b) \in \mathbb{R}^2$ such that both a and b are integers).

4. Let $P(x) = x^n + a_{n-1}x^{n-1} + \ldots + a_1 x + 1$ be a polynomial such that $a_{n-1}, \ldots, a_1$ are nonnegative real numbers. Suppose that $P(x)$ has n real roots. Prove that $P(2) \geq 3^n$.

5. Define a *neighbor* of a square S on a standard 8×8 chessboard to be any square that shares an edge with S. In each square of the chessboard, a number is placed. Suppose that, for every square S, the number placed in S is equal to the average of the numbers placed in the neighbors of S. Prove that all of the numbers placed on the chessboard are equal.

6. Let a, b, and c be complex numbers such that at least two of them are distinct. Let $z = \frac{-1}{2} + i\frac{\sqrt{3}}{2}$ and assume that $a + bz + cz^2 = 0$. Show that a, b and c are the vertices of an equilateral triangle in the complex plane.