Gordon Prize Exam

February 26, 2005

1. Let $a_1, b_1, a_2, b_2, \ldots, a_n, b_n$ be real numbers. Suppose that for all real x

$$\big|a_1 \sin(b_1 x) + a_2 \sin(b_2 x) + \cdots + a_n \sin(b_n x)\big| \le |\sin x|.$$

Prove that $|a_1 b_1 + a_2 b_2 + \cdots + a_n b_n| \le 1$.

2. Find all real solutions to $2005^x + 2002^x = 2004^x + 2003^x$.

3. Let $A = (a_{ij})_{1 \le i,j \le n}$ be an $n \times n$ matrix such that each entry a_{ij} is either 1 or -1. Prove its determinant D is divisible by 2^{n-1}.

4. Let A, B, C be the vertices of an equilateral triangle in clockwise order in the complex plane. Let X, Y, Z be the vertices of an equilateral triangle in clockwise order in the complex plane. Prove that $A + X, B + Y, C + Z$ are either the vertices of an equilateral triangle or they all coincide.

5. Prove that the first 2005 digits after the decimal point are 0 in the decimal expansion of the number $(8 + \sqrt{65}\,)^{2005}$.

6. Prove that there are infinitely many positive integers not having representation $x^2 + y^3$ where x and y are positive integers.

Answer each problem on the separate sheet containing the statement of the problem, or on an additional sheet. "Answers" without appropriate reasoning are insufficient. **Put your secret code name at the top of each answer sheet.** Please write on only one side of the answer sheets, and inside the frame.

You may take this sheet with you when you leave. Be sure to hand in separately the cover sheet (with your name, rank, student number, and secret code name).