Gordon Prize Exam

February 24, 2007

1. A positive integer is called a *palindrome* if its base-10 expansion is unchanged when it is reversed. For example, 121 and 7447 are palindromes. Show that if we denote by p_n the nth smallest palindrome, then $\sum_{n=1}^{\infty} \frac{1}{p_n}$ converges.

2. Let $z_1, ..., z_{2007}$ be equally spaced points on the unit circle. Prove that

$$\|z_1 - z_2\| \, \|z_1 - z_3\| \cdots \|z_1 - z_{2007}\|,$$

the product of the lengths of the chords from z_1 to all of the other z_k, is equal to 2007.

3. Show that integers a, b, c do not exist such that $a+b+c = -45$ and $ab+bc+ca = 9$.

4. Let W be a polynomial with real coefficients. Assume that $W(x) \geq 0$ for all real x. Prove that W may be written as a sum of squares of polynomials.

5. Let S be 6 distinct points in the plane. Let M be the maximum distance between two points of S and let m be the minimum distance between two points of S. Show that $M/m \geq \sqrt{3}$.

6. Let f_1, f_2, f_3 be linearly independent real-valued functions defined on $\mathbb{R}$. Prove that there exist $a_1, a_2, a_3 \in \mathbb{R}$ such that the matrix

$$\begin{bmatrix} f_1(a_1) & f_1(a_2) & f_1(a_3) \\ f_2(a_1) & f_2(a_2) & f_2(a_3) \\ f_3(a_1) & f_3(a_2) & f_3(a_3) \end{bmatrix}$$

is nonsingular. [Recall that functions f_1, f_2, f_3 are said to be *linearly independent* iff the only constants c_1, c_2, c_3 such that $c_1 f_1(x) + c_2 f_2(x) + c_3 f_3(x) = 0$ for all x are $c_1 = c_2 = c_3 = 0$.]

Answer each problem on the separate sheet containing the statement of the problem, or on an additional sheet. “Answers” without appropriate reasoning are insufficient. **Put your secret code name at the top of each answer sheet.**

You may take this sheet with you when you leave. Be sure to hand in separately the cover sheet (with your name, rank, student number, and secret code name).