Gordon Prize Examination

February 21, 2009

1. A spherical cherry of radius R is dropped into a glass of the form $z = (x^2 + y^2)^2$. Find the maximum R for which the cherry will reach the bottom of the glass.

2. Is there a differentiable function $f(x)$ defined for $x > 0$, satisfying $f'(x) = f(x+1)$ for all $x > 0$, and such that $\lim_{x\to\infty} f(x) = \infty$?

3. Let a and b be real numbers. Consider the power series (in powers of x) for the function $f(x) = e^{ax}\cos(bx)$. Show that the series either has no zero coefficients or has infinitely many zero coefficients.

4. Show that there is no 2009×2009 matrix A with rational entries such that $A^2 = 2I$, where I is the identity matrix.

5. Let X be the square $[0,1] \times [0,1]$ in the plane. By $|p - q|$ we will denote the distance between points $p, q \in X$. Suppose that $f\colon X \longrightarrow X$ is a surjective contraction; that is, a surjective mapping satisfying $|f(p) - f(q)| \leq |p - q|$ for all $p, q \in X$. Prove that f is actually an isometry; that is, $|f(p) - f(q)| = |p - q|$ for all $p, q \in X$.

6. Assume that your calculator is broken so that you can only add and subtract real numbers and compute their reciprocals. How can you use it to compute products?