2011 Gordon Prize examination

1. Let $1 \le a_1, \dots, a_{1006} \le 2011$ be distinct positive integers. Prove that there exist i, j such that $a_i + a_j = 2012$.

2. Let $g(x) = a_0 x^{r_0} + a_1 x^{r_1} + a_2 x^{r_2} + \ldots + a_{2011} x^{r_{2011}}$, $x > 0$, where $a_0, \dots, a_{2011}$ are nonzero real numbers and $r_0, \dots, r_{2011}$ are distinct real numbers. Prove that g has at most 2011 zeroes in $(0, \infty)$.

3. Prove that the "Pascal matrix" $P_n = \begin{pmatrix} 1 & 1 & 1 & \dots & 1 \\ 1 & 2 & 3 & \dots & n \\ 1 & 3 & 6 & \dots & \binom{n+1}{n-1} \\ \vdots & \vdots & \vdots & & \vdots \\ 1 & n & \binom{n+1}{2} & \dots & \binom{2n-2}{n-1} \end{pmatrix}$ has determinant 1.

4. Let a, b, c be distinct complex numbers. Prove that the triangle $\triangle abc$ is equilateral iff $a^2 + b^2 + c^2 = ab + bc + ca$.

5. Prove that $\displaystyle\int_{-2011}^{2011} \frac{dx}{1 + x^{2011} + \sqrt{1 + x^{4022}}} = 2011.$

6. Define $F\colon \mathbb{C}^3 \longrightarrow \mathbb{C}^3$ by $F(x, y, z) = (x + y + z, xy + yz + zx, xyz)$. Prove that F is a surjection (an "onto" map).