2012 Gordon Prize examination

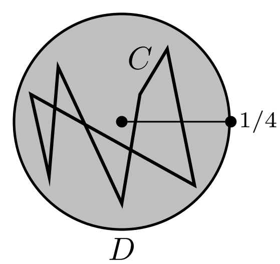
1. Let $n \in \mathbb{N}$. Find all complex number solutions $x_1, \ldots, x_n \in \mathbb{C}$ to the system of equations

$$\begin{cases} x_1 + \ldots + x_n = 0 \\ x_1^2 + \ldots + x_n^2 = 0 \\ \vdots \qquad\qquad\quad \vdots \\ x_1^n + \ldots + x_n^n = 0 \end{cases}$$

(and prove that there are no other solutions).

2. Which number is greater, $\log(5/4)$ or $\arctan(1/2)$? (Justify your answer, of course.)

3. Prove that any closed polygonal curve C of length 1 in the plane is contained in a disk D of radius 1/4. (A *polygonal curve* is a union of finitely many intervals A_1A_2, A_2A_3, ..., $A_{n-1}A_n$; it is closed if $A_n = A_1$. The statement of the problem actually holds for any closed *rectifiable* (that is, "having length") curve C, but if you don't know what it is, assume that C is polygonal.)



4. Every point of the plane is colored one of two colors, red or blue. Let R be the set of all distances between red points,

$$R = \{d(P,Q) : \text{both } P \text{ and } Q \text{ are red}\},$$

and let B be the set of all distances between blue points,

$$B = \{d(P,Q) : \text{both } P \text{ and } Q \text{ are blue}\},$$

where $d(P,Q)$ denotes the distance between points P and Q. Prove that at least one of these sets R, B is equal to $[0, \infty)$.

5. Let A and B be two (real) $n \times n$ matrices such that $A + B = AB$. Prove that $AB = BA$.

6. Let $z_1, \ldots, z_n \in \mathbb{C}$ and $|z_1| = |z_2| = \ldots = |z_n| = r > 0$. Prove that the number $a = (z_1 + z_2)(z_2 + z_3) \ldots (z_{n-1} + z_n)(z_n + z_1)(z_1 z_2 \ldots z_n)^{-1}$ is real.