2013 Gordon Prize examination problems

1. Prove that the first 2013 digits after the decimal point in the decimal expansion of the number $(6+\sqrt{37})^{2013}$ are zero.

2. Prove that for any real square matrix A, $\det(I+A^2) \geq 0$.

3. Suppose that real numbers a, b, c satisfy the equalities $\cos a+\cos b+\cos c=\sin a+\sin b+\sin c=0$. Prove that $\cos 2a+\cos 2b+\cos 2c=\sin 2a+\sin 2b+\sin 2c=0$.

4. Prove that any positive rational number can be obtained from the number 1 by applying the operations $x \mapsto x+1$ and $x \mapsto \frac{x}{x+1}$.

5. Prove that any convex polygon of area S in the plane is contained in a rectangle of area $2S$.

6. Let $f\colon \mathbb{R} \longrightarrow (0,\infty)$ be a continuous periodic function having period 1; prove that $\int_0^1 \frac{f(x)dx}{f(x+\frac{1}{2013})} \geq 1$.