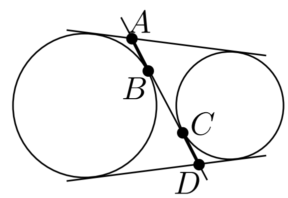
2014 Gordon examination problems

1. Prove that there does not exist a prime integer of the form $1001001\ldots1001$.

2. Let $n \in \mathbb{N}$ and suppose that S is an $(n+1)$-element subset of the set $\{1, 2, \ldots, 2n\}$. Prove that there are $a, b \in S$ (not necessarily distinct) such that the sum $a + b$ is also in S.

3. Let $p(x) = x^n + a_{n-1}x^{n-1} + \ldots + a_1 x + a_0$ be a polynomial with complex coefficients satisfying $|a_i| \leq 2014$, $i = 0, \ldots, n-1$. If $z \in \mathbb{C}$ satisfies $p(z) = 0$, prove that $|z| < 2015$.

4. The straight lines on the picture are tangent to the circles. Prove that $|AB| = |CD|$.



5. Suppose that all eigenvalues of an $n \times n$ matrix A are real and that $\mathrm{tr}(A^2) = \mathrm{tr}(A^3) = \mathrm{tr}(A^4)$. Prove that $\mathrm{tr}(A^k) = \mathrm{tr}(A)$ for all $k \in \mathbb{N}$.

6. Prove that $\displaystyle\int_0^{\pi/2} \log(\sin x)\, dx = -\frac{\pi}{2} \log 2$.