2017 Gordon examination problems

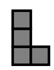
1. Find all real x satisfying the equation

$$\sqrt{\frac{x-2}{2018}}+\sqrt{\frac{x-3}{2017}}+\sqrt{\frac{x-4}{2016}}=\sqrt{\frac{x-2018}{2}}+\sqrt{\frac{x-2017}{3}}+\sqrt{\frac{x-2016}{4}}.$$

2. Suppose $z_1, \ldots, z_k$ are complex numbers of absolute value 1; for each $n = 1, 2, \ldots$ put $w_n = z_1^n + \cdots + z_k^n$. If the sequence (w_n) converges, prove that $z_1 = \cdots = z_k = 1$.

3. In an invertible $n \times n$ matrix, what is the maximal number of entries that can be equal to 1?

4. Suppose p is a polynomial with integer coefficients having at least 3 distinct integer roots. Prove that the equation $p(x) = 1$ has no integer solutions.

5. An *L-tetromino* is an L-shape made of four unit squares: . Suppose that an $m \times n$ chessboard is tiled (that is, covered completely and without overlapping) by k L-tetrominos; prove that k is even. (Each L-tetromino on the board covers 4 unit squares, and can be oriented in any way, allowing quarter-turns and reflections.)

6. For a quadratic polynomial p define the quadratic polynomials $T_1 p$ and $T_2 p$ as follows:

$$T_1 p(x) = x^2 p\big(1 + \tfrac{1}{x}\big) \text{ and } T_2 p(x) = (x-1)^2 p\big(\tfrac{1}{x-1}\big).$$

Applying the operations T_1 and T_2 in some order, is it possible to transform $x^2 + 1$ to $x^2 + 2017x + 1$?