2018 Gordon examination problems

1. Prove that for all $n \in \mathbb{N}$, $\quad 2n \sqrt[2n]{\dfrac{n!}{(3n)!}} < \log 3$.

(Here "log" stands for the natural logarithm, $\log = \ln = \log_e$.)

2. Determine whether the integer part of $(1+\sqrt{2})^{2018}$ is even or odd.

3. Let $P = A_1A_2\dots A_n$ be a regular n-gon with center O, and let $R = \operatorname{dist}(O, A_1)$. Prove that for any point X in the plane, $\sum_{k=1}^{n} \operatorname{dist}(X, A_k)^2 = n(R^2 + d^2)$, where $d = \operatorname{dist}(X, O)$.

4. Suppose an ellipse E in the plane $\mathbb{R}^2$ has no points of the lattice $\mathbb{Z}^2$ in its interior. Prove that there are at most 4 points of $\mathbb{Z}^2$ on the boundary of E.

5. Let T be a linear transformation of the vector space M_n of $n \times n$ real matrices such that $\det T(A) = \det A$ for all $A \in M_n$. Prove that T is invertible.

6. Prove that $\sin 1^\circ$ is irrational.