2020 Gordon examination problems

1. Evaluate $\displaystyle\int_{-\pi}^{\pi} \frac{\sin(2020x)}{(1+2^x)\sin x}\,dx.$

2. Let $G = \{A_1, \ldots, A_n\}$ be a finite multiplicative group of real $k \times k$ matrices and let $S = \sum_{i=1}^{n} A_i$. If $\operatorname{trace}(S) = 0$, prove that $S = 0$.

3. Find all $b \in \mathbb{N}$ for which $\sqrt[3]{2+\sqrt{b}} + \sqrt[3]{2-\sqrt{b}}$ is an integer.

4. Let $v_1, \ldots, v_n \in \mathbb{R}^d$. Prove that $\sum_{i,j=1}^{n} e^{v_i \cdot v_j} \geq n^2$ (where $u \cdot v$ denotes the inner product of vectors u and v).

5. Let $a, b \in \mathbb{R}$, $ab = 1$. Evaluate $\det \begin{pmatrix} 2 & a & a^2 & \ldots & a^{n-1} \\ b & 2 & a & \ldots & a^{n-2} \\ b^2 & b & 2 & \ldots & a^{n-3} \\ \vdots & \vdots & \vdots & & \vdots \\ b^{n-1} & b^{n-2} & b^{n-3} & \ldots & 2 \end{pmatrix}.$

6. Suppose a 10×10 board has some of its 1×1 squares colored red. After each minute passes, every non-red square that shares a side with at least two red squares also becomes red. If there are exactly 9 red squares at the start, may it happen that eventually all squares of the board become red?