Rasor-Bareis Prize Examination

February 22, 1997

1. Solve the equation $\sin^{1997} x + \cos^{1997} x = 1$.

2. Let a be a 5-digit number which does not end in 0, and let b be the same 5 digits in reverse order. Prove that $a + b$ has at least one even digit.

3. Suppose $1 < a_1 < a_2 < \cdots < a_n$ are positive integers. Prove:

$$\left(1 - \frac{1}{a_1^2}\right)\left(1 - \frac{1}{a_2^2}\right)\left(1 - \frac{1}{a_3^2}\right)\cdots\left(1 - \frac{1}{a_n^2}\right) > \frac{1}{2}.$$

4. What is the maximum of $|z^3 + z^2 - z|$, where z is a complex number with $|z| = 1$?

5. Suppose a 4×100 rectangle is covered by 200 dominoes (1×2 rectangles). Prove that there is a horizontal or vertical line which cuts through the large rectangle without cutting through any of the dominoes.

6. Each point of a straight line is colored with one of two colors. Prove that we can find a segment of nonzero length whose endpoints and midpoint have the same color.

You may take this sheet with you.
Be sure to hand in separately the cover sheet
(with your name, rank, student number, and secret code name).
Put your secret code name at the top of each answer sheet.