Rasor-Bareis Prize Examination

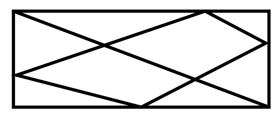
February 24, 2001

1. How many solutions are there to the equation $e^x = 20\cos x$, where $-\frac{\pi}{2} < x < 2001$? Carefully justify your answer.

2. Show that if 1001 numbers are selected from $\{1, 2, \ldots, 2001\}$ then either some two of the selected numbers differ by 1000, or some two of the selected numbers differ by 1001.

3. Prove that there is a positive integer n such that $\underbrace{11\ldots1}_{n} = \sum_{k=0}^{n-1} 10^k$ is divisible by 2001.

4. A quadrilateral Q is inscribed in a rectangle R so that each side of R contains exactly one vertex of Q. Show that the length of the perimeter of Q is at least twice that of the diagonal of R.



5. Define a *neighbor* of a square S on a standard 8×8 chessboard to be any square that shares an edge with S. In each square of the chessboard, a number is placed. Suppose that, for every square S, the number placed in S is equal to the average of the numbers placed in the neighbors of S. Prove that all of the numbers placed on the chessboard are equal.

6. Let a, b, and c be complex numbers such that at least two of them are distinct. Let $z = \frac{-1}{2} + i\frac{\sqrt{3}}{2}$ and assume that $a + bz + cz^2 = 0$. Show that a, b and c are the vertices of an equilateral triangle in the complex plane.