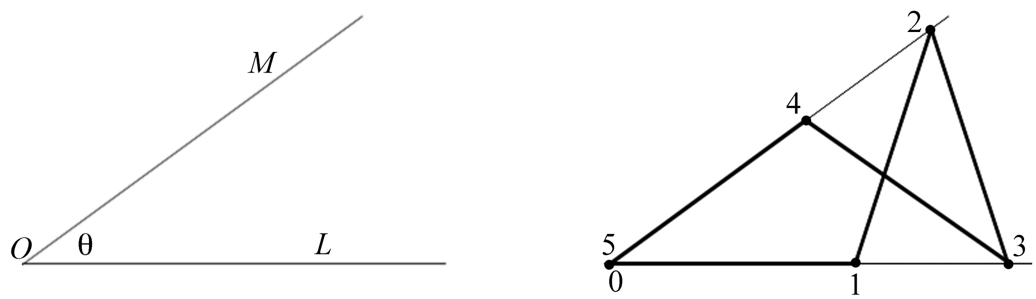
Rasor-Bareis Prize Exam

February 26, 2005

1. Suppose that the faces of a tetrahedron all have the same perimeter. Prove that the faces are all congruent.

2. Rays OL and OM emanate from point O forming angle θ. Start at O and take steps of length 1, stepping alternately from one ray to the other, and never stepping back to the preceding spot (see the picture). Suppose we arrive back at O after 5 steps. What are possible values for the angle θ?



3. What is the maximum value of the function

$$f(x) = |x|\,|x-1|\,|x-2| \cdots |x-2005| = \prod_{j=0}^{2005} |x-j|$$

for $1002 \leq x \leq 1003$?

4. Prove that 2005_b (that is, 2005 base b) is not the square of an integer for any integer base $b \geq 6$.

5. Denote by $\langle r \rangle$ the fractional part of the number r. (The *fractional part* of r is defined to be $r - \lfloor r \rfloor$, where $\lfloor r \rfloor$ is the greatest integer $\leq r$.) Prove that, given any $\epsilon > 0$, there exists a positive integer n such that

$$\left| \frac{1}{2005} - \left\langle \sqrt{n} \right\rangle \right| < \epsilon.$$

6. Prove that there are infinitely many positive integers not having representation $x^2 + y^3$ where x and y are positive integers.

Answer each problem on the separate sheet containing the statement of the problem, or on an additional sheet. "Answers" without appropriate reasoning are insufficient. **Put your secret code name at the top of each answer sheet.** Please write on only one side of the answer sheets, and inside the frame.

You may take this sheet with you when you leave. Be sure to hand in separately the cover sheet (with your name, rank, student number, and secret code name).