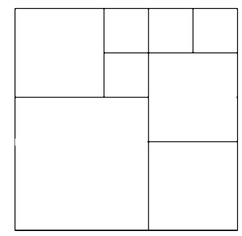
Rasor-Bareis Prize Exam

February 24, 2007

1. Define $a_0 = 1$ and $a_{n+1} = a_n/(1 + na_n)$. Determine a_{2007}.



2. Show that for any integer $n \geq 6$, a square in the plane can be dissected into exactly n squares.

3. Find

$$\lim_{n\to\infty} \left(\frac{1}{n} + \frac{1}{n+1} + \cdots + \frac{1}{2n} \right).$$

4. Show that given any 1004 elements from $\{2, 3, \ldots, 2007\}$, some two are relatively prime.

5. Determine the largest constant $k > 0$ such that for all complex numbers z_1, z_2, z_3 with $|z_1| = |z_2| = |z_3| = 1$, one has

$$|z_1 z_2 + z_2 z_3 + z_3 z_1| \geq k\,|z_1 + z_2 + z_3|.$$

6. Prove that if a parallelogram is inscribed in a circle (all four vertices on the circle), then it must be a rectangle.

Answer each problem on the separate sheet containing the statement of the problem, or on an additional sheet. "Answers" without appropriate reasoning are insufficient. **Put your secret code name at the top of each answer sheet.**

You may take this sheet with you when you leave. Be sure to hand in separately the cover sheet (with your name, rank, student number, and secret code name).