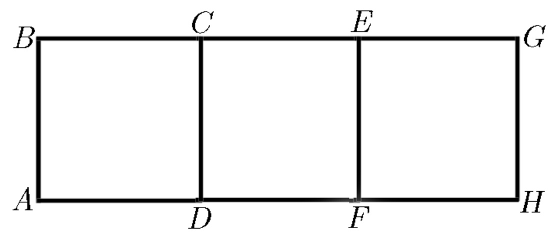
Rasor-Bareis Prize Exam

February 23, 2008

1. Solve: $\sin^{2008} x + \cos^{2008} x = 1$.

2. Let three adjacent squares be given, as in the diagram.
Show that $\angle ACB + \angle AEB + \angle AGB = 90°$.



3. Note that 2 can be written as a sum of the reciprocals of four distinct positive integers:

$$2 = \frac{1}{1} + \frac{1}{2} + \frac{1}{3} + \frac{1}{6}.$$

Can 2 be written as a sum of the reciprocals of 2008 distinct positive integers:

$$2 = \frac{1}{n_1} + \frac{1}{n_2} + \cdots + \frac{1}{n_{2008}} \qquad ?$$

4. Find all rational functions $f(x)$ such that $f(x^2 - x) = f(x^2 + x)$ for all real x.

5. Let $x_1, x_2, \cdots, x_n$ be distinct integers > 1. Prove:

$$\left(1 - \frac{1}{x_1^2}\right)\left(1 - \frac{1}{x_2^2}\right)\cdots\left(1 - \frac{1}{x_n^2}\right) > \frac{1}{2}.$$

6. Suppose $x_1 > x_2 > \ldots$ is a decreasing sequence of real numbers. Suppose

$$x_1 + \frac{x_4}{2} + \frac{x_9}{3} + \cdots + \frac{x_{n^2}}{n} < 1$$

for all n. Show that

$$x_1 + \frac{x_2}{2} + \frac{x_3}{3} + \cdots + \frac{x_n}{n} < 3.$$

Answer each problem on the separate sheet containing the statement of the problem, or on an additional sheet. "Answers" without appropriate reasoning are insufficient. **Put your secret code name at the top of each answer sheet.**

You may take this sheet with you when you leave. Be sure to hand in separately the cover sheet (with your name and secret code name).