2012 Rasor-Bareis Prize examination

1. Let $n \in \mathbb{N}$ and $a \in \mathbb{R}$. Find all real number solutions $x_1, \ldots, x_n$ to the system of equations

$$\begin{cases} x_1 + \ldots + x_n = a \\ x_1^2 + \ldots + x_n^2 = a^2 \\ \vdots \qquad\qquad\qquad \vdots \\ x_1^n + \ldots + x_n^n = a^n \end{cases}$$

(and prove that there are no other soulutions).

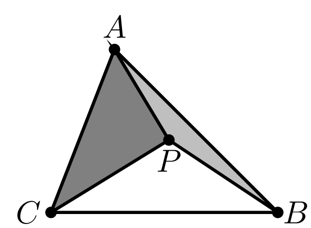
2. Let f be a real-valued function on $[0,1]$ such that $f(0) = f(1) = 0$ and

$$f\left(\frac{x+y}{2}\right) \leq f(x) + f(y) \quad \text{for all } x, y \in [0,1].$$

Prove that the set of zeroes of f is dense in $[0,1]$ (that is, for any $0 \leq a < b \leq 1$ there exists a point $x \in (a,b)$ such that $f(x) = 0$).

3. Prove that for any $x \in \mathbb{R}$, $\sin(\cos x) < \cos(\sin x)$.

4. Given a triangle $\triangle ABC$, find the set of points P inside this triangle such that $\text{area}(\triangle APC) = 2\,\text{area}(\triangle APB)$.



5. Suppose that every point of the plane is colored one of three colors, red, blue, or green. Prove that for any $x > 0$ there are points P and Q in the plane having the same color and such that $d(P,Q) = x$, where $d(P,Q)$ denotes the distance between P and Q.

6. Find all $n \in \mathbb{N}$ such that $p = \left\lfloor \frac{n^2}{3} \right\rfloor$ is prime. (For a real number x, $\lfloor x \rfloor$ denotes the integer part of x, the largest integer $\leq x$.)