1. Let $n \in \mathbb{N}$ and $a \in \mathbb{R}$. Find all real number solutions $x_1, \ldots, x_n$ to the system of equations

$$\begin{cases} x_1 + \ldots + x_n = a \\ x_1^2 + \ldots + x_n^2 = a^2 \\ \vdots \\ x_1^n + \ldots + x_n^n = a^n \end{cases}$$

(and prove that there are no other solutions).

2. Let $f$ be a real-valued function on $[0, 1]$ such that $f(0) = f(1) = 0$ and

$$f\left(\frac{x + y}{2}\right) \leq f(x) + f(y) \quad \text{for all } x, y \in [0, 1].$$

Prove that the set of zeroes of $f$ is dense in $[0, 1]$ (that is, for any $0 \leq a < b \leq 1$ there exists a point $x \in (a, b)$ such that $f(x) = 0$).

3. Prove that for any $x \in \mathbb{R}$, $\sin(\cos x) < \cos(\sin x)$.

4. Given a triangle $\triangle ABC$, find the set of points $P$ inside this triangle such that $\text{area}(\triangle APC) = 2 \times \text{area}(\triangle APB)$.

5. Suppose that every point of the plane is colored one of three colors, red, blue, or green. Prove that for any $x > 0$ there are points $P$ and $Q$ in the plane having the same color and such that $d(P, Q) = x$, where $d(P, Q)$ denotes the distance between $P$ and $Q$.

6. Find all $n \in \mathbb{N}$ such that $p = \left\lfloor \frac{n^2}{3} \right\rfloor$ is prime. (For a real number $x$, $\lfloor x \rfloor$ denotes the integer part of $x$, the largest integer $\leq x$.)