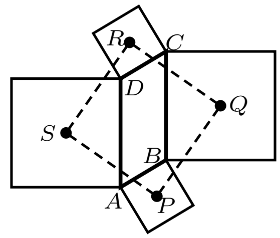
2017 Rasor-Bareis examination problems

1. Let $F_1 = 1$, $F_2 = 1$, $F_3 = 2$, $F_4 = 3, \ldots$ be the Fibonacci numbers (so that for every n, $F_{n+2} = F_n + F_{n+1}$). Show that F_n divides F_{2n} for all $n \in \mathbb{N}$.

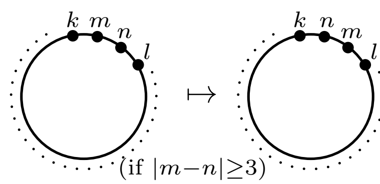
2. Let $ABCD$ be a parallelogram, and let P, Q, R, S be the centers of the squares constructed on its sides. (See the picture.) Prove that $PQRS$ is also a square.



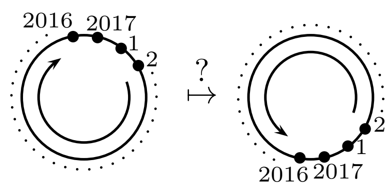
3. Prove that for each positive integer n there is a polynomial p_n of degree n with integer coefficients such that $2\cos(nx) = p_n(2\cos x)$ for all x. (For example, $p_1(y) = y$ and $p_2(y) = y^2 - 2$.)

4. Prove that there is an integer of the form $111\ldots111$ divisible by 2017.

5. If a set of integers is distributed around a circle, *a legal move* is to interchange some two neighboring integers m and n provided $|m - n| \geq 3$.



Suppose that the integers $1, 2, \ldots, 2017$ are positioned in order clockwise around the circle. Is there a sequence of legal moves that reverses the order, so that after those moves are made the integers $1, 2, \ldots, 2017$ are positioned in order counterclockwise around the circle?



6. Evaluate $\displaystyle\int_2^6 \frac{(10-x)^{2017}}{(10-x)^{2017} + (2+x)^{2017}}\, dx.$