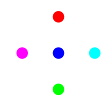
2019 Rasor-Bareis examination problems

1. Let $F_1 = 1,\ F_2 = 1,\ F_3 = 2,\ F_4 = 3,\ ...,$ be the Fibonacci sequence. Prove that for every $n \in \mathbb{N}$, $\frac{1}{F_1F_3} + \frac{1}{F_2F_4} + \cdots + \frac{1}{F_nF_{n+2}} < 1$.

2. Suppose rational numbers a and b are such that the numbers $\sqrt[3]{a} + \sqrt[3]{b}$ and $\sqrt[3]{ab}$ are also rational. Prove that $\sqrt[3]{a}$ and $\sqrt[3]{b}$ are rational as well.

3. The integer points (n, m), $n, m \in \mathbb{Z}$, of the plane $\mathbb{R}^2$ are colored with five colors so that every configuration of the form



$$\{(n, m), (n-1, m), (n+1, m), (n, m-1), (n, m+1)\},\ n, m \in \mathbb{Z},$$

contains all five colors. Prove that every length 5 horizontal row

$$\{(n, m), (n+1, m), (n+2, m), (n+3, m), (n+4, m)\},\ n, m \in \mathbb{Z}$$

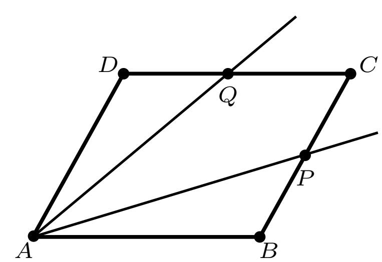
and every length 5 vertical column

$$\{(n, m), (n, m+1), (n, m+2), (n, m+3), (n, m+4)\},\ n.m \in \mathbb{Z}$$

also contain all five colors.

4. Evaluate $\int_0^{\pi/2} \frac{dx}{1+\tan^{2019} x}$.

5. Let $ABCD$ be a parallelogram and let P and Q be the midpoints of BC and of DC respectively. Is it possible that the rays AP and AQ trisect the angle $\angle BAD$ (meaning that $\angle BAP = \angle PAQ = \angle QAD$)?



6. Prove that for all $n, m \in \mathbb{N}$, $\frac{1}{\sqrt[m]{1+n}} + \frac{1}{\sqrt[n]{1+m}} \geq 1$.