2020 Rasor-Bareis examination problems

1. Prove that 17^{2020} cannot be represented as $m^3 + n^3$ for positive integers m and n.

2. Prove that for any $x, y, z \in [0, 1]$,

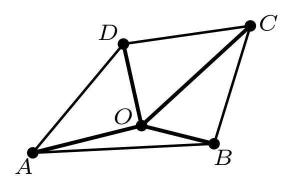
$$\frac{x}{7 + y^3 + z^3} + \frac{y}{7 + z^3 + x^3} + \frac{z}{7 + x^3 + y^3} \leq \frac{1}{3}.$$

3. Prove that $\displaystyle\int_0^{\pi/2} \cos(2020x)(\cos x)^{2018} dx = 0$.

4. Find all real polynomials $f(x) = x^{2020} + a_{2019}x^{2019} + \cdots + a_1 x + a_0$ all of whose roots are real, and such that $|f(i)| = 1$. (Here $i = \sqrt{-1}$.)

5. Let $ABCD$ be a convex quadrilateral of area 1, and let O be a point inside it. Prove that

$$|AO| + |BO| + |CO| + |DO| \geq 2\sqrt{2}.$$



6. A 6×6 board is covered with eighteen 2×1 tiles, without gaps or overlaps. No matter how those tiles are arranged, prove that there always is a straight line that cuts across the whole board without cutting any tile.