WHAT IS... MONSKY'S THEOREM?

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Abstract. We will see that if we try to dissect a square in n triangles of equal area, then n must be even. This theorem was first proven by Monsky in 1970, and we will see the proof he gave as a beautiful application of 2-adic numbers that also relies on combinatorial topology (Sperner's lemma).

1. Introduction

Suppose we have a square T in the plane and that we want to dissect it into n triangles of equal area — equiareal triangulation of T—. This is very easy to do if n is even: for instance, one can just divide the horizontal sides into $\frac{n}{2}$ segments of equal length and draw a diagonal on each of the $\frac{n}{2}$ rectangles, as in the picture. However, what if n is odd? As one may see for oneself, this already proves quite

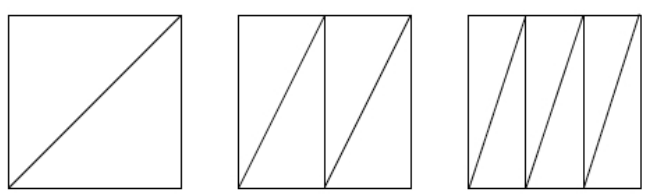

Figure 1. Dissection of the unit square I^2 for even n in the process described above.

difficult to do for even small values of odd n. One may also be led to believe by the nature of the question that this was already answered by the greek geometers thousands of years ago.

Yet, this question was first thought about by Fred Richman (1965) at New Mexico State, who was also surprised to be unable to find any reference for this. He was writing a Master's exam, and he wanted to include this question but could not solve it. He asked the question in the Am. Math. Monthly, and Paul Monsky answered it in 1970, based on initial work by John Thomas, who did it when the vertices of the triangle are rational. We state it in the following theorem, whose proof will be the object of this talk:

Theorem 1.1 (Monsky, 1970). *If a square is dissected into triangles of equal areas, then the number of triangles must be even.*

The method uses a combinatorial topological result, 2-adic valuations, and the fact that those can be extended to $\mathbb{R}$. These results are proven in the following sections, culminating in the proof given by Monsky. We conclude with some extensions and further comments.

Remark 1.2. At present, cf. [3], no other proof of this fact is known, so this theorem connects two apparently disjoint branches of math.

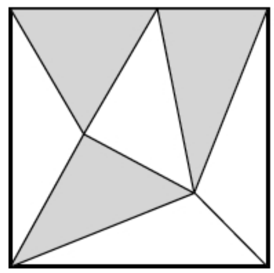

FIGURE 2. Dissection of the unit square I^2 into an odd number of triangles with almost equal areas.

2. Sperner's lemma

Let us begin with the topological lemma of Sperner.
Consider a polygon P in the plane and a triangulation of P. Color each of the vertices by one of the colors 1, 2, or 3.

Definition 2.1. We call an edge a 12-*edge* if its endpoints are colored by a 1 and a 2 respectively.

Definition 2.2. We call a triangle *complete* if the colors of each of its vertices are $1, 2, 3$ up to permutation.

Now we consider the result of Sperner, for whose proof we follow [4].

Lemma 2.3 (Sperner, 1928)**.** *Let P be a polygon whose vertices are colored by three colors $(1, 2, 3)$ let a triangulation be given for this polygon. Then the number of complete triangles is equal to the number of* 12*-edges on the boundary of the polygon* $\pmod 2$.

This lemma is used to find the existence of a complete triangle in the dissection of the square.

PROOF: We use a double counting combinatorial argument. Put a dot on each side of each 12-segment. We count the number of dots in the interior of the triangle first by noticing that each interior segment contributes either 0 or 2 dots (i.e., depending on whether it is a 12-edge), while each boundary segment contributes 0 or 1 dots. Hence, the number of dots in the interior of the triangle is equal to the number of 12-edges on the boundary of the polygon. Secondly, we count the

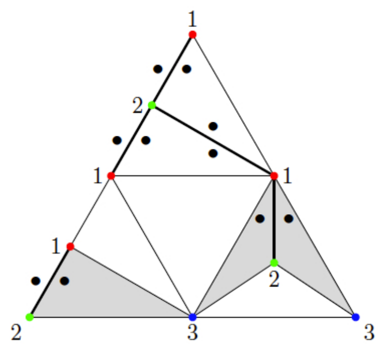

FIGURE 3. Depiction of the method of the proof of Sperner's lemma.

number of dots in the interior of each triangle in the triangulation. By construction, complete triangles contain one dot, while the rest contain an even number of dots. Thus, the number of dots is equal to the number of complete triangles (mod 2), which completes the proof. □

Remark 2.4. In [4] it is explained how Sperner's lemma is a non-trivial result in combinatorial topology, as it allows to prove Brouwer's fixed point theorem (i.e., that for the n-dimensional unit ball in $\mathbb{R}^n$, B, any continuous function $f : B \to B$ has a fixed point), in the case where $n = 2$.

3. 2-ADIC VALUATIONS AND ABSOLUTE VALUES

Now we give a brief introduction to valuations and absolute values on a field K, a key ingredient in the proof of Monsky's theorem.

Definition 3.1. Let $(\Gamma, +, <)$ be an ordered abelian group. Following [2] we say that a surjective map on a field K

$$v : K \to \Gamma \cup \{\infty\}$$

is a *valuation* if for all $x, y \in K$ we have

(1) $v(x) = \infty$ implies $x = 0$,
(2) $v(xy) = v(x) + v(y)$,
(3) $v(x + y) \geq \min\{v(x), v(y)\}$.

If $\Gamma = \{0\}$, then one has the so-called *trivial valuation.* As we will see, such a valuation provides a non-archimedean absolute value on the field $\mathbb{Q}$.

Working directly from axioms (1)-(3), we can deduce the following helpful facts about valuations. Namely, for all $x, y \in K$,

$$v(1) = 0, \quad v(x^{-1}) = -v(x), \quad v(-x) = v(x), \text{ and}$$

$$v(x) < v(y) \text{ implies } v(x+y) = v(x).$$

Definition 3.2. We say that a subring $\mathcal{O}_v$ of a field K is a *valuation ring* if for each $x \in K^\times$, either

$$x \in \mathcal{O}_v \quad \text{or} \quad x^{-1} \in \mathcal{O}_v.$$

Observe that if we have a valuation on K, then $\mathcal{O}_v := \{x \in K : v(x) \geq 0\}$ is a valuation ring of K. This valuation ring is a local ring with maximal ideal $\mathfrak{M} := \{x \in K : v(x) > 0\}$.

In the proof of Chevalley's theorem, we use the fact that a valuation on a field K is the same as a valuation ring of K. The interested reader is referred to [2].

We now give an example of a valuation of $\mathbb{Q}$ in which we are interested and show how it gives rise to the 2-adic non-archimedean absolute value.

Example 3.3. *The* 2*-adic absolute value.* We begin by defining the 2-adic absolute value on $\mathbb{Q}$ in the following way. For each $x \in \mathbb{Q}^\times$, write it uniquely as $x = 2^n \frac{a}{b}$, where a, b are odd integers and $n \in \mathbb{Z}$. Then put

$$v_2 : \mathbb{Q} \to \mathbb{Z} \cup \{\infty\} \quad \text{given by} \quad v_2(x) = v_2\left(2^n \frac{a}{b}\right) := n \quad \text{and} \quad v(0) = \infty.$$

It is straightforward to check that v_2 is a valuation on $\mathbb{Q}$ that is called *discrete* since the valuation group is $\mathbb{Z}$. In this case, the valuation ring is $\mathbb{Z}_{(2)}$, the localization at the prime ideal (2) of $\mathbb{Z}$.

Notice that so far we have not used anything special about 2, and indeed, this construction works for any prime number p, but 2 is the one we need for the proof of Monsky's theorem.

Recall that an absolute value on a field K is a function

$$|\cdot| : K \to \mathbb{R}$$

with the following properties:

(1) $|x| > 0$ for all $x \in K^\times$, and $|0| = 0$.
(2) $|xy| = |x||y|$ for all $x, y \in K$.
(3) $|x+y| \leq |x| + |y|$ for all $x, y \in K$. Moreover, we say that an absolute value is *non-archimedean* if it satisfies the ultrametric inequality:
(4) $|x+y| \leq \max\{|x|, |y|\}$ for all $x, y \in K$. Observe that this implies that if $|x| < |y|$, then $|x+y| = |y|$.

Resuming our previous example, we define the 2-adic non-archimedean absolute value on $\mathbb{Q}$. Namely, put

$$|\cdot|_2 \colon \mathbb{Q} \to \mathbb{R} \quad \text{defined by} \quad |x|_2 := 2^{-v_2(x)}, \text{ where } v_2(x) \text{ is as above.}$$

Given the properties of v_2, especially (3), it is easy to check that $|\cdot|_2$ is a non-archimedean absolute value, because $|n|_2 \leq 1$ for all $n \in \mathbb{Z}$. Observe that when $n \in \mathbb{Z}$, $|n|_2 < 1$ if and only if n is even, which is the fact we need.

Finally, let us just comment that what we need for Monsky's theorem is an extension of $v_2(x)$ to $\mathbb{R}$, which allows us to define an extension of $|\cdot|_2$ to $\mathbb{R}$ in the natural way, which we show shortly.

4. Chevalley's Theorem on extensions of valuations

In this section we provide a proof of the fact that we can extend the 2-adic absolute value of $\mathbb{Q}$ to $\mathbb{R}$. The proof of this fact is an easy corollary of Chevalley's theorem, which we prove in the following. As stated above, we will use the fact that valuation on a field K and a valuation ring of K are interchangeable.

Theorem 4.1 (Chevalley)**.** *Let K be a field, $R \subseteq K$ a subring, and $\mathfrak{p}$ a prime ideal of R. Then there exists a valuation ring $\mathcal{O}$ of K such that*

$$R \subseteq \mathcal{O} \quad \textit{and} \quad \mathfrak{M} \cap R = \mathfrak{p},$$

where $\mathfrak{M}$ is the maximal ideal of $\mathcal{O}$.

Proof: Recall that $R_\mathfrak{p}$ stands for the localization of R at $\mathfrak{p}$. Consider the set

$$\Sigma := \{(A, I) : R_\mathfrak{p} \subseteq A \subseteq K,\ \mathfrak{p}R_\mathfrak{p} \subseteq I \subsetneq A, A \text{ a ring},\ I \text{ an ideal of } A\}.$$

We want to use Zorn's lemma, so first observe that $\Sigma \neq \emptyset$ since $(R_\mathfrak{p}, \mathfrak{p}R_\mathfrak{p}) \in \Sigma$. Partially order Σ with $\leq$ by declaring $(A_1, I_1) \leq (A_2, I_2)$ iff $A_1 \subseteq A_2$ and $I_1 \subseteq I_2$.

Observe that each chain has an upper bound in $(\Sigma, \leq)$ simply by taking unions for both rings and ideals at the same time, by the usual arguments. Then, we apply Zorn's lemma to get a maximal element in Σ, which we will call $(\mathcal{O}, \mathfrak{M})$. First notice that $\mathfrak{M}$ is a maximal ideal, because otherwise, if $\mathfrak{M}'$ strictly contains $\mathfrak{M}$, then $(\mathcal{O}, \mathfrak{M}')$ contradicts the maximality of $(\mathcal{O}, \mathfrak{M})$. Moreover, $\mathcal{O}$ is a local ring since if there was another maximal ideal $\mathfrak{M}''$ in $\mathcal{O}$, then we could localize $\mathcal{O}$ at $\mathfrak{M}''$, and then the pair $(\mathcal{O}_{\mathfrak{M}''}, \mathfrak{M}''\mathcal{O}_{\mathfrak{M}''})$ would again contradict the maximality of $(\mathcal{O}, \mathfrak{M})$.

Furthermore, since by construction $R \subseteq R_\mathfrak{p} \subseteq \mathcal{O}$, we have that $\mathfrak{M} \cap R_\mathfrak{p} = \mathfrak{p}R_\mathfrak{p}$, because $\mathfrak{p}R_\mathfrak{p}$ is the maximal ideal of $R_\mathfrak{p}$. This implies that $\mathfrak{M} \cap R = \mathfrak{p}$ by standard facts about localizations of rings at a prime ideal. So it only remains to show that $\mathcal{O}$ is a valuation ring, since we already showed it is local.

For the sake of contradiction, suppose $\mathcal{O}$ is not a valuation ring, so that there exists $x \in K^\times$ such that $x, x^{-1} \notin \mathcal{O}$. Since this is the case, $\mathcal{O} \subsetneq \mathcal{O}[x], \mathcal{O}[x^{-1}]$. Again by maximality of $(\mathcal{O}, \mathfrak{M})$ we must have that $\mathfrak{M}[x] = \mathcal{O}[x]$ and $\mathfrak{M}[x^{-1}] = \mathcal{O}[x^{-1}]$. Therefore, we can find $a_0, \ldots, a_n, b_0, \ldots, b_m \in \mathfrak{M}$ such that

$$1 = \sum_{i=0}^{n} a_i x^i \quad \text{and} \quad 1 = \sum_{i=0}^{m} b_i x^{-i}$$

with n, m minimal. Suppose that $m \leq n$. Since $b_0 \in \mathfrak{M}$ and $\mathcal{O}$ is local, we have

$$\sum_{i=1}^{m} b_i x^{-i} = 1 - b_0 \in \mathcal{O}^\times = \mathcal{O} \setminus \mathfrak{M}.$$

Hence,

$$1 = \sum_{i=1}^{m} c_i x^{-i}, \quad \text{where} \quad c_i = \frac{b_i}{1 - b_0} \in \mathfrak{M}.$$

Multiplying this equation by x^n yields

$$x^n = \sum_{i=1}^{m} c_i x^{n-i}.$$

But then, using our hypothesis on the a_i's, we can write

$$1 = \sum_{i=0}^{n} a_i x^i = \sum_{i=0}^{n-1} a_i x^i + \sum_{i=1}^{m} c_i a_n x^{n-i},$$

so since $m \leq n$, we get a contradiction with the minimality of n. The case $n \leq m$ is similar. □

Let K_2/K_1 be a field extension, and $\mathcal{O}_1 \subseteq K_1$, $\mathcal{O}_2 \subseteq K_2$ be valuation rings. We say that $\mathcal{O}_2$ is a *prolongation* of $\mathcal{O}_1$ if $\mathcal{O}_2 \cap K_1 = \mathcal{O}_1$. This is denoted by $(K_1, \mathcal{O}_1) \subseteq (K_2, \mathcal{O}_2)$. We can also refer to this by saying $\mathcal{O}_2$ is an *extension* of $\mathcal{O}_1$.

Let $(K_1, \mathcal{O}_1) \subseteq (K_2, \mathcal{O}_2)$, and let $\mathfrak{M}_1$, $\mathfrak{M}_2$ be the maximal ideals of $\mathcal{O}_1$ and $\mathcal{O}_2$ respectively. Then we have

$$\mathfrak{M}_2 \cap K_1 = \mathfrak{M}_2 \cap \mathcal{O}_1 = \mathfrak{M}_1,$$

$$\mathcal{O}_2^\times \cap K_1 = \mathcal{O}_2^\times \cap \mathcal{O}_1 = \mathcal{O}_1^\times.$$

This holds because both rings are valuation rings. We also have that if $K_1 \subseteq K_2$ is a field extension and $\mathcal{O}_2$ is a valuation ring of K_2, one also sees that $\mathcal{O}_1 = \mathcal{O}_2 \cap K_1$ is a valuation ring of K_1 such that $(K_1, \mathcal{O}_1) \subseteq (K_2, \mathcal{O}_2)$. With this result we can now prove the promised extension of a valuation in the following

Theorem 4.2. *Let K_2/K_1 be a field extension, and let $\mathcal{O}_1 \subseteq K_1$ be a valuation ring. Then there is an extension of $\mathcal{O}_2$ of $\mathcal{O}_1$ in K_2.*

Proof: Since $\mathcal{O}_1$ is a subring of K_2, by Chevalley's Theorem there exists a valuation ring $\mathcal{O}_2$ of K_2 with $\mathcal{O}_1 \subseteq \mathcal{O}_2$ and $\mathfrak{M}_2 \cap \mathcal{O}_1 = \mathfrak{M}_1$ for their respective maximal ideals. Finally, since $\mathcal{O}_2 \cap K_1$ and $\mathcal{O}_1$ are valuation rings with the same maximal ideal, by the above considerations, they must coincide: one contains the other and they are both valuation rings. □

From this follows the claim we made about extending the 2-adic valuation of $\mathbb{Q}$ to $\mathbb{R}$, so the same goes for the 2-adic absolute value, as we will use in the proof of Monsky's theorem in the next section.

5. Proof of Monsky's Theorem

Now that we have all the tools required, we can proceed to give the proof of Monsky's theorem. Fix an extension of the 2-adic absolute value on $\mathbb{R}$, which by abuse of notation we call $|\cdot|_2$, which we know to exist by Chevalley's theorem. Next, partition $\mathbb{R}^2$ in the following three sets, which we can view as a coloring with colors $1, 2$, and 3:

$$S_1 := \{(x, y) : |x|_2 < 1, |y|_2 < 1\},$$

$$S_2 := \{(x, y) : |x|_2 \geq 1, |x|_2 \geq |y|_2\},$$

$$S_3 := \{(x, y) : |y|_2 \geq 1, |y|_2 > |x|_2\}.$$

To check that they are indeed a partition of $\mathbb{R}^2$, a sketch may help. Moreover, by axiom (4) of a non-archimedean absolute value we observe that points with colors 2 and 3 are translation-invariant under points with color 1. We proceed with a lemma on complete triangles in $\mathbb{R}^2$ for this coloring:

Lemma 5.1. *Let T be a triangle in $\mathbb{R}^2$ complete with respect to the above coloring. Then its area A satisfies*

$$|A|_2 > 1.$$

Proof: Since the coloring is translation-invariant, we may move our triangle T to the origin (another point of type 1), which is nothing but a translation by a point of type 1, meaning that the triangle is still complete, and its area is unaffected. Therefore, without loss of generality, suppose that T has vertices $(0,0)$, (x_2, y_2), and (x_3, y_3), where the labels indicate the coloring of each vertex (and the origin is of color 1 as mentioned).

Then we can write A as a determinant with the following formula:

$$A = \frac{1}{2}\begin{vmatrix} x_2 & x_3 \\ y_2 & y_3 \end{vmatrix} = \frac{x_2y_3 - x_3y_2}{2}.$$

But now, by our choice of coloring we have $|x_2|_2 \geq |y_2|_2$ and $|y_3|_2 > |x_3|_2$, so that $|x_2y_3|_2 > |x_3y_2|_2$. Consequently,

$$|A|_2 = \left|\frac{1}{2}\right|_2 |x_2y_3 - x_3y_2|_2 = 2|x_2y_3|_2 = 2|x_2|_2|y_3|_2 \geq 2,$$

which gives the result. □

Finally, we can prove Monsky's theorem.

Theorem 5.2 (Monsky, 1970)**.** *Let S be a square in $\mathbb{R}^2$, and suppose that we dissect it into m triangles of equal area. Then m is even.*

Proof: Without loss of generality, we may translate and dilate the square so that S is $[0,1] \times [0,1]$. Let T be a triangulation of S into m triangles with equal area. Color their vertices according to the partition given above (S_1, S_2 and S_3). Notice that on the boundary of the square $\mathrm{bd}(S)$, 12-edges only occur on the edge connecting $(0,0)$ and $(1,0)$. This holds because on the edge $(0,0)-(0,1)$ we have no points with color 2, as $|x|_2 = 0$ always. On the edge $(1,0)-(1,1)$ we have no points with color 1, because $|x|_2 = 1$ always, and on the edge $(0,1)-(1,1)$, we have no points of color 1, because $|y|_2 = 1$ always.

Moreover, on the $(0,0)-(1,0)$ edge there are no vertices of color 3 because $|y|_2 = 0$, so only 12-edges occur on this edge. Moreover, since $(0,0)$ has color 1 and $(1,0)$ has color 2, we have that the number of 12-edges is given by the number of changes from 1 to 2. It follows that in this configuration, this number must be odd, so there is an odd number of 12-edges on $\mathrm{bd}(S)$.

Therefore, by Sperner's lemma applied to the square S, there must be a complete triangle in the dissection T. By the previous lemma, we have that its area A is such that $|A|_2 > 1$, but the area of S is $mA = 1$. Since $|\cdot|_2$ is multiplicative, we must have $|m|_2 < 1$, but since m is an integer, this means that m is even. This concludes the proof. □

Remark 5.3. In Monsky's original paper, a strengthening of this result is proven. Namely in [1] it is shown that if $[0,1] \times [0,1]$ is dissected into m triangles T_i with area of T_i equal to a_i, then there is a polynomial $f \in \mathbb{Z}[x_1, \ldots, x_m]$ such that $f(a_1, \ldots, a_m) = \frac{1}{2}$.

6. Further extensions and comments

We finish by remarking that in [3] one can find a proof that is not as nonconstructive as the one we followed in [4]. Moreover, from [4] we have the following results of a similar nature:

(1) Partitioning the n-dimensional cube into simplices yields that the number of simplices must be a multiple of $n!$.
(2) Partitioning regular n-gons for $n > 4$ implies that the number of triangles is divisble by n.
(3) In 1990 Monsky showed that for a centrally symmetric polygon, the answer is the same as for the square.
(4) There are some polygons that cannot be dissected into triangles of equal areas. An example is the trapezoid with vertices $(0, 0)$, $(0, 1)$, $(1, 0)$ and $(a, 1)$, where a is not algebraic.

The same article [4], discusses related open problems.

References

1. Monsky, P.; *On Dividing a Square Into Triangles* The American Mathematical Monthly, Vol 77, No. 2, (Feb. 1970), pp. 161-164.
2. Prestel, A., Engler, A.J.; *Valued Fields* Springer Monographs in Mathematics, 2005.
3. Aigner, M., Ziegler, G. M.; *Proofs from THE BOOK* Fifth Edition, Springer, 2014, pp. 151-158.
4. Xu, M.; *Sperner's Lemma* URL: `https://math.berkeley.edu/~moorxu/misc/equiareal.pdf`