Four Color Theorem

The four-color theorem states that any map in a plane can be colored using four-colors in such a way that regions sharing a common boundary (other than a single point) do not share the same color.
This problem is sometimes also called Guthrie's problem after F. Guthrie, who first conjectured the theorem in 1852.

To simplify a map one associates a simple planar graph G to the given map, namely one puts a vertex in each region of the map, then connects two vertices with an edge if and only if the corresponding regions share a common border. The problem is then translated into a graph coloring problem: one has to paint the vertices of the graph so that no edge has endpoints of the same color.
In 1878 Cayley published the first paper on the subject. This was followed by false proofs given independently by Kempe (1879)* and Tait (1880). The proof of the 4CT was finally obtained by Appel and Haken (1977), who constructed a computer-assisted proof.

This proof required 1200 hours of computation, and introduced a collection of 1476 reducible configurations. However, because part of the proof consisted of an exhaustive analysis of many discrete cases by a computer, some mathematicians do not accept it. A shorter, independent proof was constructed by Neil Robertson (OSU), Daniel P. Sanders, Paul Seymour and Robin Thomas (1996)

This proof took 3.5 hours run time and less than one megabyte of RAM.They decreased the size of the unavoidable set to 633 and simplified the discharging method to 32 discharging rules, instead of the 300+ of Appel and Haken. Their methods went so far as to remove decimals and use only integer arithmetic to ensure that rounding error would not be an issue.

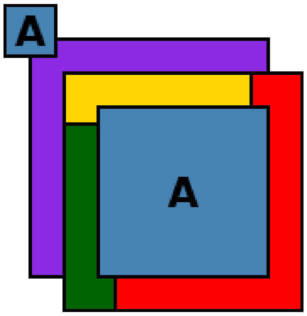
In the paper itself the authors note: "However, an argument can be made that our "proof" is not a proof in the traditional sense, because it contains steps that can never be verified by humans. ... Apart from this hypothetical possibility of a computer consistently giving an incorrect answer, the rest of our proof can be verified in the same way as traditional mathematical proofs...We concede, however, that verifying a computer program is much more difficult than checking a mathematical proof of the same length."

In 2001, the same authors announced an alternative proof, by proving the snark theorem. This proof remains largely unpublished though. Finally in 2005, the theorem was proven by Georges Gonthier with general purpose theorem proving software.

A snark is a connected, bridgeless cubic graph with chromatic index equal to 4. Somewhat more simply a graph that can not be 3 colored, whose vertices all connected to three edges. Tutte was the first one to notice that the snark theorem implies the 4CT.

There have been conjectures that 4CT is not solvable in satisfactory way by humans. That there is something so deep to this problem that we cannot break it down to few enough cases to be checkable by hand.

Despite the motivation from coloring political maps of countries, the theorem is not of particular interest to mapmakers. “Books on cartography and the history of mapmaking do not mention the four-color property.” - Kenneth May (math historian)



In 4CT we require that the regions are simply connected, removing these borken region issues. In this case for maps in which more than one country may have multiple disconnected regions, six or more colors might be required.

Five color Theorem:
This proof of the five color theorem is based on a failed attempt at the four color proof by Alfred Kempe in 1879.

First note that 3F ≤ 2E (where E = # edges and F = # faces) because all faces must have at least three edges. Since F = 2 + E − V by Euler’s Theorem, we conclude that the average degree of a vertex is

2E / V ≤ 6 − 12 / V. Thus there must be at least one vertex of degree ≤ 5. Given that we can color graphs with 5 or fewer vertices with 5 colors. Let us proceed inductively. Given a graph G with an arbitrary number of vertices, Find a vertex of degree 5 or less, and call it v.

Now remove v from G. The graph G' obtained this way has one fewer vertex than G, so we can assume by induction that it can be colored with only five colors. v must be connected to five other vertices, since if not it can be colored in G with a color not used by them. So now look at those five vertices v_1, v_2, v_3, v_4, v_5 that were adjacent to v in cyclic order (which depends on how we write G). If we did not use all the five colors on them, then obviously we can paint v in a consistent way to render our graph 5-colored.So we can assume that WLOG v_1, v_2, v_3, v_4, v_5 are colored with colors 1, 2, 3, 4, 5 respectively.

Consider the subgraph G_{13} of G' consisting of the vertices that are colored with colors 1 and 3 only, and edges connecting two of them. If v_1 and v_3 lie in different connected components of G_{13}, we can reverse the coloration of v_1.

If on the contrary v_1 and v_3 lie in the same connected component of G_{13}, we can find a path in G_{13} joining them, that is a sequence of edges and vertices painted only with colors 1 and 3. Consider the subgraph G_{24} of G', and apply the same arguments as before. Then either we are able to reverse a coloration on a subgraph of G_{24} or we can connect v_2 and v_4 with a path containing vertices colored only with colors 2 and 4. The latter possibility is a contradiction, as such a path would intersect the path we constructed in G_{13}. So G can in fact be five-colored.

Higher genus surfaces:

Heawood conjecture of Ringel–Youngs theorem gives a lower bound for the number of colors that are necessary for graph coloring on a surface of a given genus. It was formulated in 1890 by Percy John Heawood and proven in 1968 by Gerhard Ringel and Ted Youngs.

$$\gamma(g) = \left\lfloor \frac{7 + \sqrt{1 + 48g}}{2} \right\rfloor,$$